AN ENTROPY FORMULA FOR A NON-SELF-AFFINE MEASURE WITH APPLICATION TO WEIERSTRASS-TYPE FUNCTIONS

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Abstract. Let $\tau : [0,1] \to [0,1]$ be a piecewise expanding map with full branches. Given $\lambda : [0,1] \to (0,1)$ and $g : [0,1] \to \mathbb{R}$ satisfying $\tau'\lambda > 1$, we study the Weierstrass-type function

$$\sum_{n=0}^{\infty} \lambda^n(x)\, g(\tau^n(x)),$$

where $\lambda^n(x) := \lambda(x)\lambda(\tau(x))\cdots\lambda(\tau^{n-1}(x))$. Under certain conditions, Bedford proved in [3] that the box counting dimension of its graph is given as the unique zero of the topological pressure function

$$s \mapsto P((1-s)\log\tau' + \log\lambda).$$

We give a sufficient condition under which the Hausdorff dimension also coincides with this value. We adopt a dynamical system theoretic approach which is used among others in [12] and [1] to investigate special cases including the classical Weierstrass functions. For this purpose we prove a new Ledrappier-Young entropy formula, which is a conditional version of Pesin's formula, for non-invertible dynamical systems. Our formula holds for all lifted Gibbs measures on the graph of the above function, which are generally not self-affine.

1. Introduction

1.1. **Motivation and preceding results.** Let $(I_i)_{i=0}^{\ell-1}$ be a partition of $[0,1]$ into intervals with I_i° and $\overline{I_i}$ being the interiors and closures, respectively. Then we consider the map $\tau : [0,1] \to [0,1]$ such that the restrictions $\tau_{|I_i^\circ} : I_i^\circ \to (0,1)$ are C^{1+}-diffeomorphisms with $\inf(\tau_{|I_i^\circ})' > 1$ for $i \in \{0,\ldots,\ell-1\}$, where C^{1+} means that a Hölder continuous derivative exists, without specifying the Hölder exponent. In addition, let $\lambda : [0,1] \to (0,1)$ and $g : [0,1] \to \mathbb{R}$ be maps which are C^{1+} on each I_i° satisfying $\lambda\tau' > 1$.

We study the Weierstrass-type function

$$W_{\tau,\lambda}(x) := \sum_{n=0}^{\infty} \lambda^n(x) g(\tau^n(x))$$

from a dimension theoretic point of view, where $\lambda^n(x) := \lambda(x)\lambda(\tau(x))\cdots\lambda(\tau^{n-1}(x))$.

We recall a pair of selected results from the literature. To be precise, we assume during the citation that τ, λ and g can be extended to C^{1+}-functions on $\mathbb{R}/\mathbb{Z}$. Let $s(\tau,\lambda) \in \mathbb{R}$ be the unique zero of the Bowen equation

$$P((1-s)\log\tau' + \log\lambda) = 0 \tag{1}$$

and $\nu_{\tau,\lambda} \in \mathcal{P}([0,1])$ the equilibrium measure, where P denotes the topological pressure. Moreover, let $h_{\nu_{\tau,\lambda}}$ denote the associated KS-entropy.

First, the box counting dimension of the graph of $W_{\tau,\lambda}$ is proved by T. Bedforld in [3] to be

$$s(\tau,\lambda) = 1 + \frac{h_{\nu_{\tau,\lambda}} + \int \log\lambda \, d\nu_{\tau,\lambda}}{\int \log\tau' \, d\nu_{\tau,\lambda}}, \tag{2}$$

whenever the function $W_{\tau,\lambda}$ is not differentiable. Note that he generalised a result of J. Kaplanan, J. Mallet-Pareta and J. Yorkea in [9] for a significantly larger class.

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Second, A. Moss and C. P. Walkden constructed in [18] a randomized version of the function for a large class of g including trigonometric functions, in a similar fashion to B. Hunt's work [8], for which the Hausdorff and the box counting dimension of the graph coincide with the same $s(\tau, \lambda)$ almost surely.

These results lead to the conjecture that both dimensions may always be identical. Nevertheless, this general conjecture is denied by a counterexample which M. Urbanski and F. Przytycki constructed in [19] for a so-colled limit Rademacher function related to a Pisot number, making use of some combinatoric properties of that algebraic number.

Our Theorems 2, 3, 5 give a partial positive answer to this problem by providing sufficient conditions. We employ a dynamical system theoretic approach making use of the new dimension and entropy formula stated in Theorem 1.

Remark 1.1. Similar approaches were taken among others in [9], [19] and [12]. More recently, K. Barański, B. Bárány and J. Romanowska studied in [1] the classical Weierstrass function, which is given by choosing $\tau(x) = \ell x \bmod 1$, $g(x) = \cos(2\pi x)$ and λ to be constant on each I_i. Combining the results of [12] and [22], they discovered an explicit parameter region, for which the Hausdorff and the box counting dimension of the graph coincide.[1] Our Theorem 5 extends their results.

Remark 1.2. We also mention that an alternative shorter proof for the work [1] is given by G. Keller in [10], which can also be modified for our Theorems 4, 5 by using large deviation results. This is, however, not the content of this note.

1.2. **Main results.** We present our main results, some of whose proofs can be found in Sections 3, 4 and 5. Given $\xi \in [0,1]$, the function $q_\xi : [0,1] \to \mathbb{R}$ is defined by $q_\xi(x) := \pi_\xi^{ss}(x, W(x))$, where $\pi_\xi^{ss} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is the projection on the the hyperplane $\{0\} \times \mathbb{R}$ along the strong stable fibres with respect to ξ which will be introduced in Subsection 2.2 after a suitable dynamical system is constructed. The precise definitions are listed at the beginning of Section 3. Then we can disintegrate[2] the lift $\mu \in \mathcal{P}([0,1] \times \mathbb{R})$ of a τ-invariant $\nu \in \mathcal{P}([0,1])$ on the graph of $W_{\tau,\lambda}$ as

$$\mu := (\mathrm{Id}, W_{\tau,\lambda})^* \nu = \int \mu_{(\xi,y)} \, d\left(\nu \circ q_\xi^{-1}\right)(y)$$

for each $\xi \in [0,1]$, where $\mu_{(\xi,y)} \in \mathcal{P}([0,1] \times \mathbb{R})$ is to be interpreted as the conditional measure on the strong stable fibre through $(\xi, 0, y)$. Moreover, ν can be extended to $\nu^{\mathrm{ext}} \in \mathcal{P}([0,1]^2)$ naturally as (14).

Theorem 1 (Dimension and entropy formula). *Suppose that $\nu \in \mathcal{P}([0,1])$ is a Gibbs measure. Then μ, $\mu_{(\xi, q_\xi(x))}$ and $\nu \circ q_\xi^{-1}$ are exact dimensional and the dimensions are constant for ν^{ext}-a.a. $(\xi, x) \in [0,1]^2$, satisfying $\dim_H(\mu) = \dim_H\left(\mu_{(\xi,q_\xi(x))}\right) + \dim_H\left(\nu \circ q_\xi^{-1}\right)$ and*

$$h_\nu = \dim_H\left(\mu_{(\xi,q_\xi(x))}\right) \cdot \int \log \tau' \, d\nu - \dim_H\left(\nu \circ q_\xi^{-1}\right) \cdot \int \log \lambda \, d\nu.$$

Remark 1.3. The dimension and entropy formula were originally introduced by F. Ledrappier and L.-S. Young in [14] for diffeomorphisms on a compact Riemannian manifold. Then Ledrappier established in [12] their counterparts for non-invertible models through an invertible extension. He sketched the proof for the case $\tau(x) = 2x \bmod 1$ and Lebesgue measure, which we extend in Section 3.

Remark 1.4. Bárány proved in [4] the corresponding formulas for self-affine measures. Although the measures we investigate are not self-affine, some of his results are fairly similar to ours. Instead of the self-affinity, we make use of not only the Gibbs property of the marginal measures but also the structure of the function $W_{\tau,\lambda}$.

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[1] Meanwhile a result which covers all parameters was published, see [21]. He modified the sufficient condition provided in [22].

[2] See Proposition 3.3.

With additional information, the Hausdorff dimension of μ can be derived from the above formulas. Here the random variable $\Theta(\cdot, x) : [0,1] \to \mathbb{R}$, which will be defined in (11), describes the 'random' strong stable direction at the point $(x, W_{\tau,\lambda}(x))$ in sense of (12) together with (11). Note that the marginal measure $\nu^- := \nu^{\text{ext}}(\,\cdot\, \times [0,1])$ satisfies the relation (15).

Theorem 2. *Suppose that $\nu \in \mathcal{P}([0,1])$ is a Gibbs measure and let μ be its lift on the graph of $W_{\tau,\lambda}$. If the distribution of $\Theta(\cdot, x)$ under ν^- has Hausdorff dimension 1 for ν-a.a. $x \in [0,1]$, then we have*

$$\dim_H(\mu) = \min\left\{1 + \frac{h_\nu + \int \log\lambda\, d\nu}{\int \log\tau'\, d\nu}, \frac{h_\nu}{-\int \log\lambda\, d\nu}\right\},$$

where the first value is taken if and only if $\dim_H(\mu) \geqslant 1$.

Moreover, we have $\dim_H(\mu) \geqslant 1$ if and only if $h_\nu \geqslant -\int \log\lambda\, d\nu$.

Proof. By Theorem 1 together with Lemma 4.1 we have

$$\dim_H(\mu) = \begin{cases} 1 + \frac{h_\nu + \int \log\lambda\, d\nu}{\int \log\tau'\, d\nu} & \text{if } \dim_H(\mu) \geqslant 1 \\ \frac{h_\nu}{-\int \log\lambda\, d\nu} & \text{if } \dim_H(\mu) < 1 \end{cases}.$$

Observe that in case $\dim_H(\mu) \geqslant 1$ we have $h_\nu + \int \log\lambda\, d\nu \geqslant 0$, while in the other case we have $h_\nu + \int \log\lambda\, d\nu < 0$. Thus we can finish the proof, rewriting

$$\min\left\{1 + \frac{h_\nu + \int \log\lambda\, d\nu}{\int \log\tau'\, d\nu}, \frac{h_\nu}{-\int \log\lambda\, d\nu}\right\} = \frac{h_\nu}{\int \log\tau'\, d\nu} + \min\left\{1, \frac{h_\nu}{-\int \log\lambda\, d\nu}\right\} \cdot \left(1 + \frac{\int \log\lambda\, d\nu}{\int \log\tau'\, d\nu}\right).$$

□

Remark 1.5. Without the assumption on Θ, generally, the statement of the above theorem is by no means true since the value on the right hand side does not depend on g, although the one on the other side heavily does. However, we mention that our assumption is much stronger than the non-degenerate case in sense of [3], i.e. W is not differentiable.[3] For instance, in case g is non-zero constant and λ is piecewise[4] constant (but not trivial), $W_{\tau,\lambda}$ is not differentiable despite $\Theta \equiv 0$. In particular, the box counting dimension of the graph is still $s(\tau, \lambda)$ according to [3].

Remark 1.6. In the above proof we applied Lemma 4.1. This is a key lemma, whose prototype appears in Ledrappier's note [12] and is also referred to in [1]. Because of its importance we will deal with it in Section 4 separately. There we will give a complete proof for a slightly more general situation, which is based on some ideas in [12] and [15].

The next result is a consequence of the preceding theorem. Recall that $s(\tau, \lambda) \in \mathbb{R}$ and $\nu_{\tau,\lambda} \in \mathcal{P}([0,1])$ are defined in (1).

Theorem 3. *If the distribution of $\Theta(\cdot, x)$ under $\nu^-_{\tau,\lambda}$ has Hausdorff dimension 1 for $\nu_{\tau,\lambda}$-a.a. $x \in [0,1]$, then $\dim_H(\text{graph}(W_{\tau,\lambda})) = \dim_B(\text{graph}(W_{\tau,\lambda})) = s(\tau, \lambda)$.*

Proof. As mentioned, $\dim_H(\text{graph}(W_{\tau,\lambda})) \leqslant \dim_B(\text{graph}(W_{\tau,\lambda})) = s(\tau, \lambda)$ is proved in [3].[5] [6]

In order to prove $\dim_H(\text{graph}(W_{\tau,\lambda})) \geqslant s(\tau, \lambda)$, it suffices to show $\dim_H(\mu_{\tau,\lambda}) \geqslant s(\tau, \lambda)$ since $\dim_H(\text{graph}(W_{\tau,\lambda})) \geqslant \dim_H(\mu_{\tau,\lambda})$ by Lemma 2.10. Observe that $h_{\nu_{\tau,\lambda}} \geqslant -\int \log\lambda\, d\nu_{\tau,\lambda}$ follows from (2) as $s(\tau, \lambda) \geqslant 1$. Concerning the fact that the equilibrium measure $\nu_{\tau,\lambda}$ is a Gibbs measure, the claim follows from Theorem 2. □

Although the last theorem is a powerful tool to determine the Hausdorff dimension of the graph, the verification of the assumption on $\Theta(\cdot, x)$ is in many cases quite challenging. Finally, we give two examples. The details will be discussed in Section 5.

[3] In fact, W is either C^1 or nowhere continuous.

[4] Our 'piecewise' is always related to $(I_i^\circ)_{i=0}^{\ell-1}$.

[5] To be precise, in [3] all functions are assumed to be C^{1+} on $\mathbb{T}^1$, instead of the piecewise regularity. However, the difference is not essential in that proof. Alternatively, it is not difficult to derive the upper bound by means of the Gibbs property of $\nu_{\tau,\lambda}$, constructing Moran covers.

[6] Note also that a direct proof for $\dim_H(\text{graph}(W_{\tau,\lambda})) \leqslant s(\tau, \lambda)$ is given as proof of [18, Proposition 2.2].

Theorem 4. *Let $\gamma_0, \gamma_1 \in (0,1)$ and $a_0, a_1 \in \mathbb{R}$ satisfy $\gamma_0 a_0 \neq \gamma_1 a_1$. Suppose $l = 2$ and that τ and g are piecewise linear. Furthermore, suppose that $\lambda(x) := \gamma_i |I_i|$ and $g'(x) = a_i$ for $x \in I_i$, $i = 0, 1$. Then there is a set $E \subset \mathbb{R}$ of Hausdorff dimension 0 such that* $\dim_H(\mathrm{graph}(W_{\tau,t\lambda})) = \dim_B(\mathrm{graph}(W_{\tau,t\lambda})) = s(\tau, t\lambda)$ *for all* $t \in \left(\max\{\gamma_0, \gamma_1\}, \min\{\frac{\gamma_0}{\sqrt{|I_0|}}, \frac{\gamma_1}{\sqrt{|I_1|}}\}\right] \setminus E$.

Remark 1.7. In Theorem 4, if additionally $|I_0| = |I_1|$ and $\gamma_0 = \gamma_1$ are satisfied, and if $g(x) = \mathrm{dist}(x, \mathbb{Z})$, then $W_{\tau,t\lambda}$ are Takagi functions which are studied in [12].

Theorem 5. *Suppose $\ell \geqslant 2$ and that τ is a piecewise linear. Furthermore, suppose $g(x) := \cos(2\pi x)$ and $\lambda := (\tau')^{-\theta}$ for a $\theta \in (0,1)$. If both*

$$\frac{|I_i|}{|I_j|} < |I_j|^{\frac{-\theta}{2-\theta}} \quad (\forall i \neq j) \quad \textit{and} \tag{3}$$

$$G\left((\min_i |I_i|)^{1-\theta}, (\max_i |I_i|)^{1-\theta}\right) + G\left((\min_i |I_i|)^{2-\theta}, (\max_i |I_i|)^{2-\theta}\right) < \delta_0, \tag{4}$$

are satisfied, then $\dim_H(\mathrm{graph}(W_{\tau,\lambda})) = \dim_B(\mathrm{graph}(W_{\tau,\lambda})) = 2 - \theta$, *where*

$$\delta_0 := \inf_{i \neq j} \inf_x \sin^2\left(\pi(\rho_i(x) - \rho_j(x))\right), \quad \textit{and} \quad G(s,t) := \left(s^{-1}\left(\frac{t^2}{1-t} + \frac{t-s}{2}\right)\right)^2 \tag{5}$$

for $s, t \in (0,1)$.

Remark 1.8. Let us consider the special case of Theorem 5, where $\tau(x) := \ell x \bmod 1$. Furthermore, let $\lambda \in (1/\ell, 1)$ be a constant function. Then the condition (3) is trivial, while the other condition (4) is satisfied, if $\ell \geqslant 3$ and $(\tau')^{-\theta} = \ell^{-\theta} \in (\tilde{\lambda}_\ell, 1)$. Here $\tilde{\lambda}_\ell$ is the unique zero of

$$h_\ell(\lambda) := \frac{1}{(\ell\lambda - 1)^2} + \frac{1}{(\ell^2\lambda - 1)^2} - \sin^2\left(\frac{\pi}{\ell}\right).$$

This is a part of [1, Theorem A] which we technically extended. Notice that the cited theorem provides another condition for the case $\ell = 2$ which apparently relies on the specific choice of τ.

2. Preliminaries

As long as the parameters are fixed, we simply write W instead of $W_{\tau,\lambda}$.

Here is the general notation of this note:

- Let $\rho_i : [0,1] \to \overline{I_i}$ be the continuous extension of the inverse map of the branch $\tau|_{I_i^\circ}$ for each $i \in \{0, \ldots, \ell - 1\}$.
- Let $\rho_{(\omega_1,\ldots,\omega_n)} := \rho_{\omega_n} \circ \cdots \circ \rho_{\omega_1}$ for $(\omega_1, \ldots, \omega_n) \in \{0, \ldots, \ell-1\}^n$ and $n \in \mathbb{N}$.
- Let $k(x) := i$ for $x \in I_i$.
- Let $[x]_n := (k(x), k(\tau(x)), \ldots, k(\tau^{n-1}(x))) \in \{0, \ldots, \ell-1\}^n$ for $x \in [0,1]$ and $n \in \mathbb{N}$.
- Let $I_N(x)$ be the monotonicity interval of τ^N containing x for $x \in [0,1]$ and $N \in \mathbb{N}$.
- Let $\mathcal{S}_\tau := \{I_N(x) : x \in [0,1] \text{ and } N \in \mathbb{N}\} \cup \{\emptyset\}$.
- For a function ϕ on $[0,1]$ we write $\phi^n := \prod_{i=0}^{n-1} \phi \circ \tau^i$ and $\phi_n := \sum_{i=0}^{n-1} \phi \circ \tau^i$.
- We use expressions like τ', when the non-differential points can be ignored.

For a metric space E let $\mathcal{P}(E)$ be the set of Borel probability measures and $\mathcal{B}(E)$ be the Borel algebra. Further, for a family $\mathcal{F} \subseteq \mathcal{B}(E)$ let $\sigma(\mathcal{F})$ denote the smallest σ-algebra containing $\mathcal{F}$. Note that $\mathcal{S}_\tau$ is a semiring, which generates $\mathcal{B}([0,1])$. Thus if two finite Borel measures on $[0,1]$ have identical values on $\mathcal{S}_\tau$, they must be the same measure. Indeed, we have the following theorem.

Lemma 2.1 (Corollary of the approximation theorem (cf. [11, Theorem 1.65(ii)])). *For any $\nu \in \mathcal{P}([0,1])$, $A \in \mathcal{B}([0,1])$ and $\varepsilon > 0$ there are $N \in \mathbb{N}$ and mutually disjoint $I_1, \ldots, I_N \in \mathcal{S}_\tau$ such that*

$$\nu\left(A \triangle \bigcup_{i=1}^N I_i\right) < \varepsilon.$$

2.1. **The dynamical system and its inverse.** We introduce a dynamical system which characterises the graph of $W_{\tau,\lambda}$ as a repeller. Then we construct an invertible extension which enables us to work with an attractor.

If we consider the skew product dynamical system $G : [0,1] \times \mathbb{R} \to [0,1] \times \mathbb{R}$ defined by $G(x,y) := \left(\tau(x), \frac{y-g(x)}{\lambda(x)}\right)$, the graph of W can be characterised as its unique repeller. Indeed, the graph is an invariant set, i.e. we have

$$G(x, W(x)) = (\tau(x), W(\tau(x)))$$

for all $x \in [0,1]$, while all other points are driven to infinity because of the uniform expansion in the vertical direction.

Now, we construct an inverse system of this as follows. First, we extend the basis dynamics $\tau : [0,1] \to [0,1]$ to the second component of the inverse of the following non-linear Baker map. We define the non-linear Baker map $B : [0,1]^2 \to [0,1]^2$ by

$$B(\xi, x) := (\tau(\xi), \rho_{k(\xi)}(x)).$$

Notice that our definition of the Baker map may be unusual, especially when τ preserves Lebesgue measure. We just adopt the simplest one. Observe that $B^{-1}(\xi, x) = (\rho_{k(x)}(\xi), \tau(x))$. Then we consider the skew product system $F : [0,1]^2 \times \mathbb{R} \to [0,1]^2 \times \mathbb{R}$ by

$$F(\xi, x, y) := \left(B(\xi, x),\, \lambda\left(\rho_{k(\xi)}(x)\right) \cdot y + g\left(\rho_{k(\xi)}(x)\right)\right).$$

Clearly, the second and third coordinates of F is the inverse to G. Moreover, it is straightforward to verify that

$$F(\xi, x, W(x)) = \left(B(\xi, x), W\left(\rho_{k(\xi)}(x)\right)\right) \tag{6}$$

holds for all $(\xi, x) \in [0,1]$, i.e. the graph of $(\xi, x) \mapsto W(x)$ is invariant under F. Moreover, this graph is the global attractor of the system due to the uniform contraction in the vertical direction.

Observe that any iterates are also skew-products, i.e. we can define for $n \in \mathbb{N}$ the fibre map $F^n_{(\xi,x)}(y) : \mathbb{R} \to \mathbb{R}$ with respect to F^n by

$$F^n(\xi, x, y) =: \left(B^n(\xi, x), F^n_{(\xi,x)}(y)\right).$$

Indeed, one can calculate

$$F^n_{(\xi,x)}(y) = \lambda^n\left(\rho_{[\xi]_n}(x)\right) \cdot y + W_n\left(\rho_{[\xi]_n}(x)\right), \tag{7}$$

where $W_n(x) := \sum_{j=0}^{n-1} \lambda^j(x)\, g(\tau^j x)$.

We define the (KS-)entropy w.r.t. a τ-invariant probability measure $\nu \in \mathcal{P}([0,1])$ by

$$h_\nu := -\lim_{N\to\infty} \frac{1}{N} \int \log \nu(I_N(x))\, d\nu(x). \tag{8}$$

Recall that, if ν is ergodic, then the Shannon-McMillan-Breiman theorem guarantees the convergence

$$h_\nu = \lim_{N\to\infty} \frac{-\log \nu(I_N(x))}{N} \tag{9}$$

for ν-a.a. $x \in [0,1]$.

We also note the following facts.

Proposition 2.2. *If $\nu \in \mathcal{P}([0,1])$ is ergodic, then we have*

$$\lim_{N\to\infty} \frac{-\log |I_N(x)|}{N} = \int \log \tau'\, d\nu$$

for ν-a.a. $x \in [0,1]$. In particular, $\int \log \tau'\, d\nu = -\int \log |I_{k(\cdot)}|\, d\nu$.

Proof. By the mean-value theorem and the distortion estimate of $\log \tau'$ there is a $C > 0$ such that

$$C^{-1} \leqslant (\tau^N)'(x) \cdot |I_N(x)| \leqslant C$$

for all $x \in [0,1]$ and $N \in \mathbb{N}$. Now, the claim follows by Birkhoff ergodic theorem. □

The following elementary property of the function W describes its local behaviour accurately.

Proposition 2.3 ([18, Proposition 3.1][7])**.** *There is a $C_m > 0$ such that*

$$\sup_{u,v\in I_N(x)} |W(u) - W(v)| \leqslant C_m\, \lambda^N(x)$$

holds for all $x \in [0,1]$ and $N \in \mathbb{N}$.

2.2. **Strong stable fibres.** Observe that F has the derivative matrix

$$DF(\xi,x,y) = \begin{pmatrix} \tau'(\xi) & 0 & 0 \\ 0 & (1/\tau')(\rho_{k(\xi)}(x)) & 0 \\ 0 & ((y\lambda' + g')/\tau')(\rho_{k(\xi)}(x)) & \lambda(\rho_{k(\xi)}(x)) \end{pmatrix},$$

where we consider only those $x \in [0,1]$ for which all derivatives exist.

As this is a triangular matrix, one can see in its diagonal the characteristic contracting and expanding scales of this uniformly hyperbolic system. Especially, the middle value contributes to the strongest contraction. In order to determine the corresponding direction, which we call the strong stable direction, we define

$$X_3(\xi,x,y) \;:=\; -\sum_{n=1}^{\infty} \gamma^n\left(\rho_{[\xi]_n}(x)\right)\left(F^{n-1}_{(\xi,x)}(y)\cdot\lambda'\left(\rho_{[\xi]_n}(x)\right) + g'\left(\rho_{[\xi]_n}(x)\right)\right), \text{ and} \tag{10}$$

$$\Theta(\xi,x) \;:=\; X_3(\xi,x,W(x)) \tag{11}$$

for $(\xi,x,y) \in [0,1]^2 \times \mathbb{R}$, where $\gamma(x) := 1/(\tau'\lambda)(x)$. Since we have

$$DF(\xi,x,y)\begin{bmatrix} 0 \\ 1 \\ X_3(\xi,x,y)\end{bmatrix} = \frac{1}{\tau'\left(\rho_{k(\xi)}(x)\right)}\begin{bmatrix} 0 \\ 1 \\ X_3\circ F(\xi,x,y)\end{bmatrix} \tag{12}$$

for each $(\xi,x,y) \in [0,1]^2 \times \mathbb{R}$, the vector $(0,1,X_3(\xi,x,y))^T$ indicates the strong stable direction at (ξ,x,y). More precisely, let $\ell^{ss}_{(\xi,x,y)}$ be the solution of the initial vale problem

$$\begin{cases} \left(\ell^{ss}_{(\xi,x,y)}\right)'(v) = X_3\left(\xi, v, \ell^{ss}_{(\xi,x,y)}(v)\right) \\ \ell^{ss}_{(\xi,x,y)}(x) = y \end{cases}.$$

Clearly, the curve $v \mapsto \left(\xi, v, \ell^{ss}_{(\xi,x,y)}(v)\right)$ represents the strong stable fibre through (ξ,x,y), which in particular satisfies

$$F\left(\xi, v, \ell^{ss}_{(\xi,x,y)}(v)\right) = \left(B(\xi,v), \ell^{ss}_{F(\xi,x,y)}\left(\rho_{k(\xi)}(v)\right)\right). \tag{13}$$

Slightly abusing notation, we call the function $\ell^{ss}_{(\xi,x,y)}$ also the strong stable fibre through (ξ,x,y).

The next proposition shows that the strong stable fibres are almost parallel to each other.

Proposition 2.4. *There is a $C_s > 0$ such that*

$$C_s^{-1}\,|y-y'| \leqslant \left|\ell^{ss}_{(\xi,x,y)}(v) - \ell^{ss}_{(\xi,x,y')}(v)\right| \leqslant C_s\,|y-y'|$$

holds for all $\xi,x,v \in [0,1]$, $y,y' \in \mathbb{R}$ and $r > 0$.

Proof. From (7) and (10) follows

$$\begin{aligned} \ell^{ss}_{(\xi,x,y)}(v) - \ell^{ss}_{(\xi,x,y')}(v) &= y - y' + \int_x^v X_3\left(\xi, s, \ell^{ss}_{(\xi,x,y)}(s)\right) - X_3\left(\xi, s, \ell^{ss}_{(\xi,x,y')}(s)\right) ds \\ &= y - y' - \int_x^v A(\xi,s)\,\left(\ell^{ss}_{(\xi,x,y)}(s) - \ell^{ss}_{(\xi,x,y')}(s)\right) ds, \end{aligned}$$

where

$$A(\xi,s) := \sum_{n=1}^{\infty} \gamma^n\left(\rho_{[\xi]_n}(s)\right)\,\lambda^{n-1}\left(\rho_{[\xi]_{n-1}}(s)\right)\,\lambda'\left(\rho_{[\xi]_n}(s)\right).$$

[7] Note that the additional assumption of that paper that all functions are C^{1+} on $\mathbb{T}^1$ is not used in the proof of [18, Proposition 3.1].

Thus we have

$$\ell^{ss}_{(\xi,x,y)}(v) - \ell^{ss}_{(\xi,x,y')}(v) = (y-y')\, e^{-\int_x^v A(\xi,s)\,ds} \in [e^{-\|A\|_\infty}, e^{\|A\|_\infty}] \cdot |y-y'|.$$

□

Any τ-invariant $\nu \in \mathcal{P}([0,1])$ has the unique B-invariant extension $\nu^{\mathrm{ext}} \in \mathcal{P}([0,1]^2)$ which is determined by

$$\nu^{\mathrm{ext}}\left(I_M\left(\rho_{[\xi]_M}(x)\right) \times I_N(x)\right) := \nu\left(I_{M+N}\left(\rho_{[\xi]_M}(x)\right)\right) \tag{14}$$

for $M, N \in \mathbb{N}$ and $(\xi, x) \in [0,1]^2$.

Furthermore, we define the (τ-invariant) marginal measure $\nu^- := \nu^{\mathrm{ext}}(\,\cdot\, \times [0,1])$. Then there is a conditional distribution with respect to the vertical partition $\{\{\xi\} \times [0,1] : \xi \in [0,1]\}$ which can be written as product measures $\delta_\xi \otimes \nu_\xi^+$ with $\nu_\xi^+ \in \mathcal{P}([0,1])$ and satisfies

$$\nu^{\mathrm{ext}} = \int \delta_\xi \otimes \nu_\xi^+ \, d\nu^-(\xi), \tag{15}$$

see [6, Example 5.16]. In addition, let $\mu^{\mathrm{ext}} := (\mathrm{Id}, W)^* \nu^{\mathrm{ext}}$ and $\mu_\xi^+ := (\mathrm{Id}, W)^* \nu_\xi^+$.

Remark 2.5. The preceding disintegration of the measure is a prototypical example of Rokhlin's work [20] (see also [6, Theorem 5.14]). We mention that some statements in Section 3 can be alternatively verified by applying this theorem.

2.3. **Gibbs measure.** We call a τ-invariant $\nu \in \mathcal{P}([0,1])$ Gibbs measure for the potential ϕ which is Hölder continuous on I_i for each $i \in \{0, \ldots, \ell-1\}$, if there are $C_\phi > 0$ and $P(\phi) \in \mathbb{R}$ such that

$$\frac{1}{C_\phi} \leqslant \frac{\nu(I_N(x))}{e^{\phi_N(x) - NP(\phi)}} \leqslant C_\phi \tag{16}$$

for all $x \in [0,1]$ and $N \in \mathbb{N}$.

Note that our potential is always supposed to be Hölder continuous in the above sense.

We prove several properties of Gibbs measures. In this subsection, let ν be a Gibbs measure.

Proposition 2.6. *We have*

$$C_\phi^{-3} \leqslant \frac{\nu\left(I_N(x) \cap \tau^{-N} A\right)}{\nu(I_N(x)) \cdot \nu(A)} \leqslant C_\phi^3$$

for all $x \in [0,1]$, $N \in \mathbb{N}$ and $A \in \mathcal{B}([0,1])$.

Proof. Let $I_N(x), I_M(\tilde{x}) \in \mathcal{S}_\tau$ be arbitrary. Observe that $I_N(x) \cap \tau^{-N}\left(I_M(\tilde{x})\right) = I_{N+M}(x')$ and $I_M(\tau^N x') = I_M(\tilde{x})$ for any $x' \in I_N(x) \cap \tau^{-N} I_M(\tilde{x})$. Since such a x' always exists, by (16) we have

$$\begin{aligned}
\nu\left(I_N(x) \cap \tau^{-N}\left(I_M(\tilde{x})\right)\right) &\in [C_\phi^{-1}, C_\phi] \cdot e^{\phi_{N+M}(x') - (N+M)P(\phi)} \\
&\subseteq [C_\phi^{-2}, C_\phi^2] \cdot e^{\phi_N(x') - NP(\phi)} \cdot \nu(I_M(\tau^N x')) \\
&\subseteq [C_\phi^{-3}, C_\phi^3] \cdot \nu(I_N(x)) \cdot \nu(I_M(\tilde{x})).
\end{aligned}$$

As this consequence is true for all $I_M(\tilde{x}) \in \mathcal{S}_\tau$, the claim follows by Lemma 2.1. □

Proposition 2.7. *We have*

$$\frac{\nu_\xi^+}{\nu}, \frac{\mu_\xi^+}{\mu} \in [C_\phi^{-3}, C_\phi^3]$$

for ν^--a.a. ξ.

In particular, we have

$$\frac{\nu^{\mathrm{ext}}}{\nu^- \otimes \nu} \in [C_\phi^{-3}, C_\phi^3],$$

where $\nu^- \otimes \nu$ denotes the product measure.

Proof. Consider the σ-algebras $\mathcal{I}_N := \sigma\left(\{I_N(\xi) \times [0,1] : \xi \in [0,1]\}\right)$ for $N \in \mathbb{N}$. Observe that the filtration $(\mathcal{I}_N)_{N\in\mathbb{N}}$ has the limit σ-algebra $\mathcal{I}_\infty := \sigma(\bigcup_N \mathcal{I}_N) = \mathcal{B}([0,1]) \times [0,1]$. Let $\tilde{x} \in [0,1]$ and $M \in \mathbb{N}$ be fixed. Then, by (14) and (16) we have

$$\begin{aligned} \frac{\nu^{\mathrm{ext}}(I_N(\xi) \times I_M(\tilde{x}))}{\nu^{\mathrm{ext}}(I_N(\xi) \times [0,1])} &= \frac{\nu(I_{M+N}(\rho_{[\xi]_N}(\tilde{x})))}{\nu(I_N(\rho_{[\xi]_N}(\tilde{x})))} \\ &\in [C_\phi^{-2}, C_\phi^2] \cdot \frac{e^{\phi_{M+N}(\rho_{[\xi]_N}(\tilde{x}))-(M+N)P(\phi)}}{e^{\phi_N(\rho_{[\xi]_N}(\tilde{x}))-NP(\phi)}} \\ &\subseteq [C_\phi^{-3}, C_\phi^3] \cdot \nu(I_M(\tilde{x})). \end{aligned}$$

Furthermore, since $(\delta_\xi \otimes \nu_\xi^+)_{\xi\in[0,1]}$ is a conditional distribution of ν^{ext} with respect to $\mathcal{I}_\infty$, by the martingale convergence theorem we obtain

$$\begin{aligned} \nu_\xi^+(I_M(\tilde{x})) &= \nu^{\mathrm{ext}}([0,1] \times I_M(\tilde{x})|\mathcal{I}_\infty)(\xi, x) \\ &= \lim_{N\to\infty} \nu^{\mathrm{ext}}([0,1] \times I_M(\tilde{x})|\mathcal{I}_N)(\xi, x) \\ &= \lim_{N\to\infty} \frac{\nu^{\mathrm{ext}}(I_N(\xi) \times I_M(\tilde{x}))}{\nu^{\mathrm{ext}}(I_N(\xi) \times [0,1])} \quad \in [C_\phi^{-3}, C_\phi^3] \cdot \nu(I_M(\tilde{x})) \end{aligned}$$

for ν^{ext}-a.a. (ξ, x). As this consequence is true for all $I_M(\tilde{x}) \in \mathcal{S}_\tau$, the inclusion of ν_ξ^+/ν in the first claim follows by Lemma 2.1, which implies also the inclusion of μ_ξ^+/μ. Finally, the second claim now follows by virtue of (15). □

Proposition 2.8. *Let $t \in (0,1)$. Then, for ν-a.a. $x \in [0,1]$ there is a $N_x \in \mathbb{N}$ such that*

$$B_{|I_N(x)|t^N}(x) \subseteq I_N(x)$$

holds for all $N \geqslant N_x$.

Proof. We introduce the symbolic notation $I_{\omega_1,\ldots,\omega_N} := I_{\omega_1} \cap \tau^{-1}(I_{\omega_2}) \cap \cdots \cap \tau^{-(N-1)}(I_{\omega_N})$ for $\omega_1, \ldots, \omega_N \in \{0, \ldots, \ell-1\}$. Furthermore, let

$$E_{M,N} := \bigcup_{\omega_1,\ldots,\omega_N\in\{0,\ldots,\ell-1\}} I_{(\omega_1,\ldots,\omega_N)*0_M} \cup I_{(\omega_1,\ldots,\omega_N)*1_M},$$

where $*$ is the concatenation operator, and 0_M and 1_M are the sequences of 0 and 1 with length M, respectively. Let $\delta := \min\{|I_0|, |I_{\ell-1}|\}$. Then, for $\alpha := \frac{\log t}{\log \delta}$ let $M_N := \lfloor \alpha(N-1) \rfloor$ and $E_N := E_{M_N,N}$. Note that $\phi(0) < P(\phi)$ and $\phi(1) < P(\phi)$ are satisfied due to the Gibbs property, as 0 and 1 are fix points of τ. Since

$$\nu(E_N) = \nu([0_{M_N}]) + \nu([1_{M_N}]) \leqslant C_\phi \left(e^{((N-1)\alpha-1)(\phi(0)-P(\phi))} + e^{((N-1)\alpha-1)(\phi(1)-P(\phi))}\right)$$

is summable, by Borel-Cantelli Lemma for ν-a.e. x there is a N_x such that $x \notin E_N$ holds for all $N \geqslant N_x$, which means especially that

$$B_{|I_N(x)|t^N}(x) = B_{|I_N(x)|\delta^{N\alpha}}(x) \subseteq B_{|I_N(x)|\delta^{M_N+\alpha}}(x) \subseteq I_N(x).$$

□

2.4. **Dimension of measures.** We review some facts from the dimension theory. For a metric space E the lower and upper pointwise dimension of $\mu \in \mathcal{P}(E)$ at $x \in E$ is defined by

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu\left(B_r(x)\right)}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu\left(B_r(x)\right)}{\log r},$$

where $B_r(x)$ denotes the closed ball with radius $r > 0$ and centre $x \in E$. If the limit exists, we can also define the pointwise dimension

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu\left(B_r(x)\right)}{\log r}.$$

Moreover, the measure μ is called exact dimensional, if $d_\mu(x)$ exists and is constant for μ-a.a. $x \in E$. In this case, we call $\dim_H(\mu) := d_\mu$ the Hausdorff dimension of μ.

We give slightly more flexible forms.

Proposition 2.9. *Let μ be a Borel measure on a metric space E. Let $K > 0$ and $(\beta_N)_{N\in\mathbb{N}} \subset (0,\infty)$ be such that $\beta_N \searrow 0$ monotonically by $N \to \infty$ and*

$$\lim_{N\to\infty} \frac{\log \beta_{N+1}}{\log \beta_N} = 1.$$

Then we have

$$\underline{d}_\mu(x) = \liminf_{N\to\infty} \frac{\log \mu\left(B_{K\beta_N}(x)\right)}{\log \beta_N} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{N\to\infty} \frac{\log \mu\left(B_{K\beta_N}(x)\right)}{\log \beta_N}$$

for each $x \in E$.

Proof. Let $x \in E$ be fixed. The claim follows from the fact that

$$\frac{\log \mu\left(B_{K\beta_N}(x)\right)}{\log K\beta_{N+1}} \leqslant \frac{\log \mu\left(B_r(x)\right)}{\log r} \leqslant \frac{\log \mu\left(B_{K\beta_{N+1}}(x)\right)}{\log K\beta_N}$$

holds for all $r > 0$ and $N \in \mathbb{N}$ such that $K\beta_{N+1} \leqslant r < K\beta_N$. □

In this note, for a given (τ-invariant or not) $\nu \in \mathcal{P}([0,1])$ we study lift $\mu \in \mathcal{P}([0,1]\times\mathbb{R})$ on the graph of W, i.e. $\mu := (\mathrm{Id}, W)^*\nu$. This lifted measure μ plays a crucial roll since its lower pointwise dimension delivers the lower bound of the Hausdorff dimension of the graph of W as the next lemma shows.

Lemma 2.10. *Let μ be the lift of a $\nu \in \mathcal{P}([0,1])$ on W. If $\underline{d}_\mu \geqslant d$ μ-a.s. for a $d \in \mathbb{R}$, then $\dim_H(\mathrm{graph}(W)) \geqslant d$.*

Proof. As $\mu\left(\mathrm{graph}(W)\right) = 1$, the claim follows from [2, Theorem 2.1.5]. □

In order to calculate pointwise dimensions making use of the underlying dynamics, we often need to deal with bad-shaped objects instead of balls. For this purpose, the following general version of Besicovitch covering theorem, which is originally proved in [17], is a very powerful tool.

Lemma 2.11 (Besicovitch covering theorem (cf. [5, P.6 Remarks (4)]))**.** *Let A be a bounded subset of $\mathbb{R}^n$. For each $x \in A$ a set $H(x)$ is given satisfying the two following properties:* (a) *there exists a fixed number $M > 0$, independent of x, and two closed Euclidean balls centered at x, $B_{r(x)}(x)$ and $B_{Mr(x)}(x)$ such that $B_{r(x)}(x) \subseteq H(x) \subseteq B_{Mr(x)}(x)$;* (b) *for each $z \in H(x)$, the set $H(x)$ contains the convex hull of the set $\{z\} \cup B_{r(x)}(x)$. Then one can select from among $(H(x))_{x\in A}$ a sequence $(H_k)_k$ satisfying*

1. *The set A is covered by the sequence, i.e. $A \subseteq \bigcup_k H_k$.*
2. *No point of $\mathbb{R}^n$ is in more than $\theta_{n,M}$ (a number that only depends on n, M) elements of the sequence $(H_k)_k$.*
3. *The sequence $(H_k)_k$ can be divided in $\xi_{n,M}$ (a number that depends only on n, M) families of disjoint elements.*

As the first application, we derive a flexible version of Lebesgue density theorem.

Lemma 2.12 (Borel density theorem)**.** *Let $\nu \in \mathcal{P}(\mathbb{R}^n)$ and $g \in L^1_\nu$. Suppose that for all $x \in \mathbb{R}^n$ and $\delta > 0$ there is a measurable set $H_\delta(x)$ such that:*

- *$(H_\delta(x))_{x\in\mathbb{R}^n}$ satisfies (a) and (b) of Lemma 2.11 (with $M > 0$ independent of δ), and*
- *$\lim_{\delta\to 0} \mathrm{diam}(H_\delta(x)) = 0$ for each $x \in \mathbb{R}^n$.*

Then we have

$$\lim_{\delta\to 0} \frac{1}{\nu(H_\delta(x))} \int_{H_\delta(x)} g \, d\nu = g(x)$$

for ν-a.a. x.

Proof. As this result is surely not new, we only sketch the proof mimicking that of [13, Lemma 4.1.2]. Since the claim is clearly true for continuous g, it suffices to show that the set of the functions

g which satisfy the above condition is norm closed in L^1_ν. The closedness follows immediately if we show the maximal inequality

$$\mu\left\{x\in[0,1]:\sup_{\delta>0}\frac{1}{\nu(H_\delta(x))}\int_{H_\delta(x)} g\,d\nu>\lambda\right\}\leqslant\frac{\xi_{n,M}}{\lambda}\int g\,d\nu,$$

for $\lambda>0$, as we demonstrate at the end of the proof of Proposition 3.13 for a slightly different situation.

Finally, the proof of the maximal inequality is the same as that of [13, Lemma 4.1.1(a)] if we replace the classical Besicovitch covering theorem by Lemma 2.11. □

3. Ledrappier-Young theory

We state and prove several formulas about the entropies and dimensions. The goal is to prove Theorem 1.

In this section we use the following convention.

- Let $\nu\in\mathcal{P}([0,1])$ be a Gibbs measure for a potential ϕ.
- Let C_ϕ denote the constant of the Gibbs measure ν.
- Let $\mu\in\mathcal{P}([0,1]\times\mathbb{R})$ be the lift of ν on the graph of $x\mapsto W(x)$.
- Let $\nu^{\text{ext}}\in\mathcal{P}([0,1]^2)$ be the B-invariant extension of ν.
- Let $\mu^{\text{ext}}\in\mathcal{P}([0,1]^2\times\mathbb{R})$ be the lift of ν^{ext} on the graph of $(\xi,x)\mapsto W(x)$.
- Let $I_{(\omega_1,\ldots,\omega_N)}:=I_{\omega_1}\cap\tau^{-1}I_{\omega_2}\cdots\cap\tau^{-N-1}I_{\omega_N}$ for $\omega_1,\ldots,\omega_N\in\{0,\ldots,\ell-1\}$ and $N\in\mathbb{N}$.
- Let $R_{(\omega_1,\ldots,\omega_N)}:=I_{(\omega_1,\ldots,\omega_N)}\times\mathbb{R}$ and $R_N(x):=I_N(x)\times\mathbb{R}$.
- Let $Z_{(\omega_1,\ldots,\omega_N)}:=\left\{(\eta_i)_{\mathbb{N}}\in\{0,\ldots,\ell-1\}^{\mathbb{N}}:\eta_1=\omega_1,\ldots,\eta_N=\omega_N\right\}$ be the cylinder set.
- Let $\mathcal{Z}:=\left\{Z_{(\omega_1,\ldots,\omega_N)}:\omega_1,\ldots,\omega_N\in\{0,\ldots,\ell-1\}\text{ and }N\in\mathbb{N}\right\}\cup\{\emptyset\}$.
- Let $\chi:\{0,\ldots,\ell-1\}^{\mathbb{N}}\to[0,1]$ be the coding defined by $\bigcap_{N\in\mathbb{N}}\overline{I_{(\omega_1,\ldots,\omega_N)}}=\{\chi(\omega)\}$.
- Let $\pi^{ss}_\xi(x,y):=\ell^{ss}_{(\xi,x,y)}(0)$ and $\Pi^{ss}(\xi,x,y):=(\xi,\pi^{ss}_\xi(x,y))$.
- Let $q_\xi(x):=\pi^{ss}_\xi(x,W(x))$ and $q(\xi,x):=(\xi,q_\xi(x))$.
- Let $B^T_{\xi,r}(y):=(\pi^{ss}_\xi)^{-1}([y-r,y+r])$.
- Let $B^T_r(\xi,y):=(\Pi^{ss})^{-1}([\xi-r,\xi+r]\times[y-r,y+r])$.
- Let $\Sigma_r(\xi,x):=\left\{(v,y)\in[0,1]\times\mathbb{R}:|\ell^{ss}_{(\xi,v,y)}(x)-W(x)|\leqslant r\right\}$.
- Let δ_x denote the Dirac measure on a point x.

In addition, we simply write $B_r(x,y)$ instead of $B_r((x,y))$.

As usual, we consider the probability measures $\mathcal{P}(\{0,\ldots,\ell-1\}^{\mathbb{N}})$ with respect to the the cylinder topology. The following fact is well-known.

Lemma 3.1 (Corollary of the extension theorem)**.** *Suppose that the set function $\zeta:\mathcal{Z}\to[0,1]$ is additive and $\zeta(\emptyset)=0$. Then there is a unique $\tilde\zeta\in\mathcal{P}(\{0,\ldots,\ell-1\}^{\mathbb{N}})$ such that $\tilde\zeta_{|\mathcal{Z}}=\zeta$.*

Proof. Evidently, the family $\mathcal{Z}$ is a semiring. As every cylinder set is compact, from the additivity follows the σ-subadditivity. Therefore we can apply the extension theorem (cf. [11, Theorem 1.53]). □

3.1. **Conditional measures for μ.** The purpose of this subsection is to define the conditional measures of μ with respect to the induced ξ-strong stable fibres.

Let $\xi\in[0,1]$ be fixed arbitrarily. Let us consider

$$\mathcal{G}_\xi:=\left\{z\in\mathbb{R}:\begin{array}{l}\lim_{r\to0}\frac{\mu\left(B^T_{\xi,r}(z)\cap R_{(\omega_1,\ldots,\omega_N)}\right)}{\mu\left(B^T_{\xi,r}(z)\right)}\text{ exists}\\ \text{for all }\omega_1,\ldots,\omega_N\in\{0,\ldots,\ell-1\}\text{ and }N\in\mathbb{N}\end{array}\right\}.$$

We have $\mu\circ(\pi^{ss}_\xi)^{-1}(\mathcal{G}_\xi)=1$ since by Lemma 2.12 we have for all $\omega_1,\ldots,\omega_N\in\{0,\ldots,\ell-1\}$ and $N\in\mathbb{N}$ that

$$\lim_{r\to\infty}\frac{\mu\left(B^T_{\xi,r}(z)\cap R_{(\omega_1,\ldots,\omega_N)}\right)}{\mu\left(B^T_{\xi,r}(z)\right)}=\rho_{\xi,(\omega_1,\ldots,\omega_N)}(z)\tag{17}$$

holds for $\mu \circ (\pi_\xi^{ss})^{-1}$-a.a. z, where $\rho_{\xi,(\omega_1,\ldots,\omega_N)}(z)$ denotes the Radon–Nikodym derivative

$$\frac{d\left(\mu\left(\left((\pi_\xi^{ss})^{-1}\cdot\right)\cap R_{(\omega_1,\ldots,\omega_N)}\right)\right)}{d(\mu\circ(\pi_\xi^{ss})^{-1})}.$$

Let $z \in \mathcal{G}_\xi$. We define

$$\tilde{\nu}_{(\xi,z)}\left(Z_{(\omega_1,\ldots,\omega_N)}\right) := \rho_{\xi,(\omega_1,\ldots,\omega_N)}(z) \tag{18}$$

for all $Z_{(\omega_1,\ldots,\omega_N)} \in \mathcal{Z}$. As this set function is clearly additive on $\mathcal{Z}$, by Lemma 3.1 we can extend it to a measure $\tilde{\nu}_{(\xi,z)} \in \mathcal{P}(\{0,\ldots,\ell-1\}^{\mathbb{N}})$ uniquely. Now, we define $\nu_{(\xi,z)} := \tilde{\nu}_{(\xi,z)} \circ \chi^{-1}$. For $z \notin \mathcal{G}_\xi$ let simply $\nu_{(\xi,z)} := \nu$. Now, we can define the family of conditional measures[8] $\{\mu_{(\xi,z)} : (\xi,z) \in [0,1]\times\mathbb{R}\}$, where

$$\mu_{(\xi,z)} := (\mathrm{Id}, \ell^{ss}_{(\xi,0,z)})^* \nu_{(\xi,z)}.$$

Proposition 3.2. *We have*

$$\int \tilde{\nu}_{(\xi,z)}\left(\{\omega\}\right)\, d\left(\mu\circ(\pi_\xi^{ss})^{-1}\right)(z) = 0$$

for any $\xi \in [0,1]$ and $\omega \in \{0,\ldots,\ell-1\}^{\mathbb{N}}$.

Proof. By the monotone convergence theorem, we have

$$\begin{aligned}
\int \tilde{\nu}_{(\xi,z)}\left(\{\omega\}\right)\, d\left(\mu\circ(\pi_\xi^{ss})^{-1}\right)(z) &= \lim_{N\to\infty}\int \tilde{\nu}_{(\xi,z)}\left(Z_{\omega_1,\ldots,\omega_N}\right)\, d\left(\mu\circ(\pi_\xi^{ss})^{-1}\right)(z)\\
&= \lim_{N\to\infty}\int \rho_{\xi,(\omega_1,\ldots,\omega_N)}(z)\, d\left(\mu\circ(\pi_\xi^{ss})^{-1}\right)(z)\\
&= \lim_{N\to\infty}\nu\left(I_{(\omega_1,\ldots,\omega_N)}\right) \quad = \quad 0,
\end{aligned}$$

where the last equality is due to the Gibbs property. □

Proposition 3.3. *Let $\xi \in [0,1]$. Then we have*

$$\mu = \int \mu_{(\xi,q_\xi(x))}\, d\nu(x).$$

In particular, for ν-a.a. x we have

$$\mu_{(\xi,q_\xi(x))}\left(\mathrm{graph}(W)\right) = 1. \tag{19}$$

Proof. Evidently, the family

$$\mathcal{C} := \left\{R_{(\omega_1,\ldots,\omega_N)}\cap(\pi_\xi^{ss})^{-1}A : \omega_1,\ldots,\omega_N \in \{0,\ldots,\ell-1\},\, N\in\mathbb{N} \text{ and } A\in\mathcal{B}(\mathbb{R})\right\}$$

is $\cap$-stable. We consider

$$V_r^{ss}(x,y) := \left\{(\tilde{x},\tilde{y}) : \tilde{x}\in I_{N_r(x)}(x) \text{ and } \left|\ell^{ss}_{(\xi,\tilde{x},\tilde{y})}(x) - y\right| \leqslant r\right\},$$

where $N_r(x) := \min\{N : |I_N(x)| \leqslant r\}$. As $V_r^{ss}(x,y) \in \mathcal{C}$ for all $(x,y)\in[0,1]\times\mathbb{R}$ and $r>0$, we have $\sigma(\mathcal{C}) = \mathcal{B}([0,1]\times\mathbb{R})$. Observe that $\chi(Z_{(\omega_1,\ldots,\omega_N)})\triangle I_{(\omega_1,\ldots,\omega_N)}$ is a finite set. Thus, by (17), (18) and Proposition 3.2 we have

$$\begin{aligned}
\mu\left(R_{(\omega_1,\ldots,\omega_N)}\cap(\pi_\xi^{ss})^{-1}A\right) &= \int_A \rho_{\xi,(\omega_1,\ldots,\omega_N)}\, d\left(\mu\circ(\pi_\xi^{ss})^{-1}\right)\\
&= \int_A \tilde{\nu}_{(\xi,z)}\left(Z_{(\omega_1,\ldots,\omega_N)}\right)\, d\left(\mu\circ(\pi_\xi^{ss})^{-1}\right)(z)\\
&= \int_A \nu_{(\xi,z)}\left(I_{(\omega_1,\ldots,\omega_N)}\right)\, d\left(\mu\circ(\pi_\xi^{ss})^{-1}\right)(z)\\
&= \int \mu_{(\xi,q_\xi(x))}\left(R_{(\omega_1,\ldots,\omega_N)}\cap(\pi_\xi)^{-1}A\right)\, d\nu(x)
\end{aligned}$$

[8] Lemma 3.4 implies that, given $\xi\in[0,1]$, the family $\{\mu_{(\xi,\pi_\xi^{ss}(x,y))} : (x,y)\in[0,1]\times\mathbb{R}\}$ is actually a conditional distribution of μ with respect to the σ-algebra of the ξ-strong stable fibres, i.e. $(\pi_\xi^{ss})^{-1}\mathcal{B}(\mathbb{R})$.

for all $R_{(\omega_1,\dots,\omega_N)} \cap (\pi_\xi)^{-1}A \in \mathcal{C}$ since $\mu_{(\xi,z)}\left((\pi_\xi^{ss})^{-1}(z)\right) = 1$. This finishes the proof of the first claim.

Finally, the remaining claim is true as $\int \mu_{(\xi,q_\xi(x))}(\mathrm{graph}(W))\, d\nu(x) = \mu(\mathrm{graph}(W)) = 1$. □

Lemma 3.4. *We have*

$$\int f(x, q_\xi(x))\, d\nu(x) = \int \left[\int f(\tilde{x}, q_\xi(x))\, d\nu_{(\xi,q_\xi(x))}(\tilde{x})\right] d\nu(x)$$

for any bounded measurable $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ and $\xi \in [0,1]$.

Proof. By Proposition 3.3 we have

$$\begin{aligned}\int f(x, q_\xi(x))\, d\nu(x) &= \int f\left(x, \pi_\xi^{ss}(x,y)\right) d\mu(x,y)\\ &= \iint f\left(\tilde{x}, \pi_\xi^{ss}\left(\tilde{x}, \ell^{ss}_{(\xi,0,q_\xi(x))}(\tilde{x})\right)\right) d\nu_{(\xi,q_\xi(x))}(\tilde{x})\, d\nu(x)\\ &= \iint f(\tilde{x}, q_\xi(x))\, d\nu_{(\xi,q_\xi(x))}(\tilde{x})\, d\nu(x).\end{aligned}$$

□

3.2. **Conditional measures for μ^{ext}.** The purpose of this subsection is to define the conditional measures of μ^{ext} with respect to the strong stable fibres on each ξ-hyperplane. We also define the associated conditional entropy.

Let us consider the set

$$\mathcal{F} := \left\{ (\xi,z) \in [0,1]\times\mathbb{R} : \begin{array}{c} \lim_{r\to 0} \frac{\mu^{\mathrm{ext}}\left(B_r^T(\xi,z)\cap[0,1]\times R_{(\omega_1,\dots,\omega_N)}\right)}{\mu^{\mathrm{ext}}(B_r^T(\xi,z))} \\ \text{exists for all } \omega_1,\dots,\omega_N \in \{0,\dots,\ell-1\} \text{ and } N\in\mathbb{N} \end{array} \right\}.$$

We have $\mu^{\mathrm{ext}} \circ (\Pi^{ss})^{-1}(\mathcal{F}) = 1$ since by Lemma 2.12 we have for all $\omega_1,\dots,\omega_N \in \{0,\dots,\ell-1\}$ and $N\in\mathbb{N}$ that

$$\lim_{r\to\infty} \frac{\mu^{\mathrm{ext}}\left(B_r^T(\xi,z)\cap[0,1]\times R_{(\omega_1,\dots,\omega_N)}\right)}{\mu^{\mathrm{ext}}\left(B_r^T(\xi,z)\right)} = \rho^{\mathrm{ext}}_{(\omega_1,\dots,\omega_N)}(\xi,z) \tag{20}$$

holds for $\mu^{\mathrm{ext}}\circ(\Pi^{ss})^{-1}$-a.a. (ξ,z), where $\rho^{\mathrm{ext}}_{(\omega_1,\dots,\omega_N)}$ denotes the Radon–Nikodym derivative

$$\frac{d\left(\mu^{\mathrm{ext}}\left(((\Pi^{ss})^{-1}\,\cdot\,)\cap[0,1]\times R_{(\omega_1,\dots,\omega_N)}\right)\right)}{d(\mu^{\mathrm{ext}}\circ(\Pi^{ss})^{-1})}.$$

Let $(\xi,z)\in\mathcal{F}$. We define

$$\tilde{\nu}^{\mathrm{ext}}_{(\xi,z)}\left(Z_{(\omega_1,\dots,\omega_N)}\right) := \rho^{\mathrm{ext}}_{(\omega_1,\dots,\omega_N)}(\xi,z) \tag{21}$$

for all $Z_{(\omega_1,\dots,\omega_N)} \in \mathcal{Z}$. As this set function is clearly additive on $\mathcal{Z}$, by Lemma 3.1 we can extend it to a measure $\tilde{\nu}^{\mathrm{ext}}_{(\xi,z)} \in \mathcal{P}(\{0,\dots,\ell-1\}^{\mathbb{N}})$ uniquely. Now, we define $\nu^{\mathrm{ext}}_{(\xi,z)} := \tilde{\nu}^{\mathrm{ext}}_{(\xi,z)} \circ \chi^{-1}$. For $(\xi,z)\notin\mathcal{F}$ let simply $\nu^{\mathrm{ext}}_{(\xi,z)} := \nu$. Now, we can define the family of conditional measures[9] $\{\mu^{\mathrm{ext}}_{(\xi,z)} : (\xi,z)\in[0,1]\times\mathbb{R}\}$ by letting

$$\mu^{\mathrm{ext}}_{(\xi,z)} := (\mathrm{Id}, \ell^{ss}_{(\xi,0,z)})^* \nu^{\mathrm{ext}}_{(\xi,z)}.$$

Proposition 3.5. *We have*

$$\int \tilde{\nu}^{\mathrm{ext}}_{(\xi,z)}(\{\omega\})\, d\left(\mu^{\mathrm{ext}}\circ(\pi_\xi^{ss})^{-1}\right)(z) = 0$$

for each $\omega \in \{0,\dots,\ell-1\}^{\mathbb{N}}$.

[9] Lemma 3.7 implies that, the family $\{\mu^{\mathrm{ext}}_{\Pi^{ss}(\xi,x,y)} : (\xi,x,y)\in[0,1]^2\times\mathbb{R}\}$ is a conditional distribution of μ^{ext} with respect to the σ-algebra of the strong stable fibres, i.e. $(\Pi^{ss})^{-1}\mathcal{B}([0,1]\times\mathbb{R})$.

Proof. By the monotone convergence theorem, we have

$$\begin{aligned}\int \tilde{\nu}^{\text{ext}}_{(\xi,z)}\left(\{\omega\}\right) d\left(\mu^{\text{ext}}\circ(\Pi^{ss})^{-1}\right)(\xi,z) &= \lim_{N\to\infty}\int \tilde{\nu}^{\text{ext}}_{(\xi,z)}\left(Z_{\omega_1,\dots,\omega_N}\right) d\left(\mu^{\text{ext}}\circ(\Pi^{ss})^{-1}\right)(\xi,z)\\ &= \lim_{N\to\infty}\int \rho^{\text{ext}}_{(\omega_1,\dots,\omega_N)}(\xi,z)\, d\left(\mu^{\text{ext}}\circ(\Pi^{ss})^{-1}\right)(\xi,z)\\ &= \lim_{N\to\infty}\nu^{\text{ext}}\left([0,1]\times I_{(\omega_1,\dots,\omega_N)}\right)\\ &= \lim_{N\to\infty}\nu\left(I_{(\omega_1,\dots,\omega_N)}\right) \quad = \quad 0,\end{aligned}$$

where the last equality is due to the Gibbs property. □

Proposition 3.6. *We have*

$$\mu^{\text{ext}} = \int \delta_\xi \otimes \mu^{\text{ext}}_{(\xi,q_\xi(x))}\, d\nu^{\text{ext}}(\xi,x).$$

Proof. Evidently, the family

$\tilde{\mathcal{C}} := \left\{[0,1]\times R_{(\omega_1,\dots,\omega_N)}\cap(\Pi^{ss})^{-1}A : \omega_1,\dots,\omega_N\in\{0,\dots,\ell-1\},\, N\in\mathbb{N}, \text{ and } A\in\mathcal{B}([0,1]\times\mathbb{R})\right\}$

is $\cap$-stable. We consider

$$\tilde{V}^{ss}_r(\xi,x,y) := \left\{(\tilde{\xi},\tilde{x},\tilde{y}) : \tilde{x}\in I_{N_r(x)}(x),\ |\xi-\tilde{\xi}|\leqslant r \text{ and } \left|\ell^{ss}_{(\tilde{\xi},\tilde{x},\tilde{y})}(x)-y\right|\leqslant r\right\},$$

where $N_r(x) := \min\{N : |I_N(x)|\leqslant r\}$. As $\tilde{V}^{ss}_r(\xi,x,y)\in\mathcal{C}$ for all $(\xi,x,y)\in[0,1]^2\times\mathbb{R}$ and $r>0$, we have $\sigma(\mathcal{C}) = \mathcal{B}([0,1]^2\times\mathbb{R})$. Observe that $\chi(Z_{(\omega_1,\dots,\omega_N)})\triangle I_{(\omega_1,\dots,\omega_N)}$ is a finite set. Thus, by (20), (21) and Proposition 3.5 we have

$$\begin{aligned}&\mu^{\text{ext}}\left([0,1]\times R_{(\omega_1,\dots,\omega_N)}\cap(\Pi^{ss})^{-1}A\right)\\ =\ &\int_A \rho^{\text{ext}}_{(\omega_1,\dots,\omega_N)}\, d\left(\mu^{\text{ext}}\circ(\Pi^{ss})^{-1}\right)\\ =\ &\int_A \tilde{\nu}^{\text{ext}}_{(\xi,z)}\left(Z_{(\omega_1,\dots,\omega_N)}\right) d\left(\mu^{\text{ext}}\circ(\Pi^{ss})^{-1}\right)(\xi,z)\\ =\ &\int_A \nu^{\text{ext}}_{(\xi,z)}\left(I_{(\omega_1,\dots,\omega_N)}\right) d\left(\mu^{\text{ext}}\circ(\Pi^{ss})^{-1}\right)(\xi,z)\\ =\ &\int_A \delta_\xi\otimes\mu^{\text{ext}}_{(\xi,z)}\left([0,1]\times R_{(\omega_1,\dots,\omega_N)}\right) d\left(\mu^{\text{ext}}\circ(\Pi^{ss})^{-1}\right)(\xi,z)\\ =\ &\int \delta_\xi\otimes\mu^{\text{ext}}_{(\xi,q_\xi(x))}\left([0,1]\times R_{(\omega_1,\dots,\omega_N)}\cap(\Pi^{ss})^{-1}A\right) d\nu^{\text{ext}}(\xi,x)\end{aligned}$$

for all $[0,1]\times R_{(\omega_1,\dots,\omega_N)}\cap(\Pi^{ss})^{-1}A\in\tilde{\mathcal{C}}$ since $\delta_\xi\otimes\mu^{\text{ext}}_{(\xi,z)}\left((\Pi^{ss})^{-1}\left((\xi,y)\right)\right)=1$. This finishes the proof. □

Lemma 3.7. *We have*

$$\int f\left(x,\xi,q_\xi(x)\right) d\nu^{\text{ext}}(\xi,x) = \int\left[\int f\left(\tilde{x},\xi,q_\xi(x)\right) d\nu^{\text{ext}}_{(\xi,q_\xi(x))}(\tilde{x})\right] d\nu^{\text{ext}}(\xi,x)$$

for all bounded measurable $f:[0,1]^2\times\mathbb{R}\to\mathbb{R}$.

Proof. By Proposition 3.6 we have

$$\begin{aligned}\int f\left(x,\xi,q_\xi(x)\right) d\nu^{\text{ext}}(\xi,x) &= \int f\left(x,\Pi^{ss}(\xi,x,y)\right) d\mu^{\text{ext}}(\xi,x,y)\\ &= \iint f\left(\tilde{x},\Pi^{ss}\left(\xi,\tilde{x},\ell^{ss}_{(\xi,0,q_\xi(x))}(\tilde{x})\right)\right) d\nu^{\text{ext}}_{(\xi,q_\xi(x))}(\tilde{x})\, d\nu^{\text{ext}}(\xi,x)\\ &= \iint f\left(\tilde{x},\xi,q_\xi(x)\right) d\nu^{\text{ext}}_{(\xi,q_\xi(x))}(\tilde{x})\, d\nu^{\text{ext}}(\xi,x).\end{aligned}$$

□

The next proposition can be interpreted as the tower rule, regarding the family $\{\mu_\xi^+ : \xi \in [0,1]\}$ as a conditional distribution w.r.t. the partition into the stable hyperplanes, i.e. $\{\{\xi\} \times [0,1] \times \mathbb{R} : \xi \in [0,1]\}$.

Proposition 3.8. *We have*

$$\nu_\xi^+ = \int \nu^{\mathrm{ext}}_{(\xi, q_\xi(x))} \, d\nu_\xi^+(x)$$

for ν^--a.a. ξ.

In particular, we have for ν^{ext}-a.a. (ξ, x) that

$$\lim_{r \to 0} \frac{\mu_\xi^+ \left(B^T_{\xi,r}(q_\xi(x)) \cap R_{(\omega_1,\dots,\omega_N)} \right)}{\mu_\xi^+ \left(B^T_{\xi,r}(q_\xi(x)) \right)} = \nu^{\mathrm{ext}}_{(\xi, q_\xi(x))} \left(I_{(\omega_1,\dots,\omega_N)} \right) \tag{22}$$

holds for all $\omega_1, \dots, \omega_N \in \{0, \dots, \ell - 1\}$ and $N \in \mathbb{N}$.

Proof. Let $\tilde{\nu}_\xi := \int \nu^{\mathrm{ext}}_{(\xi, q_\xi(x))} \, d\nu_\xi^+(x)$ for $\xi \in [0,1]$. In view of the uniqueness of the conditional distribution, for the first part of the claim it suffices to show that the family $\{\delta_\xi \otimes \tilde{\nu}_\xi : \xi \in [0,1]\}$ is a conditional distribution of ν^{ext} with respect to the vertical partition $\eta := \{\{\xi\} \times [0,1] : \xi \in [0,1]\}$ as $\{\delta_\xi \otimes \nu_\xi^+ : \xi \in [0,1]\}$ is. This is true since by (15) and Proposition 3.6 we have

$$\begin{aligned}
\int \delta_\xi \otimes \tilde{\nu}_\xi(A \times B) \, \mathbf{1}_{C \times [0,1]}(\xi, x) \, d\nu^{\mathrm{ext}}(\xi, x) &= \iint \delta_\xi(A \cap C) \cdot \nu^{\mathrm{ext}}_{(\xi, q_\xi(x))}(B) \, d\nu_\xi^+(x) \, d\nu^-(\xi) \\
&= \int \delta_\xi \otimes \mu^{\mathrm{ext}}_{(\xi, q_\xi(x))}((A \cap C) \times (B \times \mathbb{R})) \, d\nu^{\mathrm{ext}}(\xi, x) \\
&= \mu^{\mathrm{ext}}((A \cap C) \times (B \times \mathbb{R})) \\
&= \nu^{\mathrm{ext}}((A \times B) \cap (C \times [0,1]))
\end{aligned}$$

for all $A, B, C \in \mathcal{B}([0,1])$, where $\mathbf{1}_{C \times [0,1]}$ denotes the indicator function of $C \times [0,1]$. Indeed, we have $\sigma(\{A \times B : A, B \in \mathcal{B}([0,1])\}) = \mathcal{B}([0,1]^2)$ and $\sigma(\{C \times [0,1] : C \in \mathcal{B}([0,1])\}) = \sigma(\eta)$.

In order to prove the remaining part, let $\omega_1, \dots, \omega_N \in \{0, \dots, \ell - 1\}$ and $N \in \mathbb{N}$. Observe that for ν^--a.a. ξ we have that

$$\begin{aligned}
\mu_\xi^+ \left(B^T_{\xi,r}(z) \cap R_{(\omega_1,\dots,\omega_N)} \right) &= \int \mu^{\mathrm{ext}}_{(\xi, \tilde{z})} \left(B^T_{\xi,r}(z) \cap R_{(\omega_1,\dots,\omega_N)} \right) \, d\left(\nu_\xi^+ \circ q_\xi^{-1} \right)(\tilde{z}) \\
&= \int_{B_r(z)} \nu^{\mathrm{ext}}_{(\xi, \tilde{z})} \left(I_{(\omega_1,\dots,\omega_N)} \right) \, d\left(\nu_\xi^+ \circ q_\xi^{-1} \right)(\tilde{z})
\end{aligned}$$

holds for all $z \in \mathbb{R}$ and $r > 0$. Thus, for any such $\xi \in [0,1]$, the convergence of (22) holds for ν_ξ^+-a.a. x by Lemma 2.12. In view of (15) the convergence holds thus for ν^{ext}-a.a. (ξ, x). □

Proposition 3.9. *We have*

$$\frac{\nu^{\mathrm{ext}}_{(\xi, q_\xi(x))}}{\nu_{(\xi, q_\xi(x))}} \in [C_\phi^{-6}, C_\phi^6]$$

for ν^{ext}-a.a. (ξ, x).

Proof. From (17), (18) and (22) together with Proposition 2.7 we have for ν^{ext}-a.a. (ξ, x) that

$$\nu_{(\xi, q_\xi(x))} \left(I_{(\omega_1,\dots,\omega_N)} \right) \in [C_\phi^{-6}, C_\phi^6] \cdot \nu^{\mathrm{ext}}_{(\xi, q_\xi(x))} \left(I_{(\omega_1,\dots,\omega_N)} \right)$$

holds for all $I_{(\omega_1,\dots,\omega_N)} \in \mathcal{S}_\tau$. Thus the claim follows by Lemma 2.1. □

Proposition 3.10. *We have*

$$\nu^{\mathrm{ext}}_{q \circ B^{-1}(\xi, x)} \left(I_N(\tau(x)) \right) = \frac{\nu^{\mathrm{ext}}_{q(\xi, x)} \left(I_{N+1}(x) \right)}{\nu^{\mathrm{ext}}_{q(\xi, x)} \left(I_1(x) \right)}$$

for all $N \in \mathbb{N}$ and ν^{ext}-a.a. (ξ, x).

Proof. Let $\eta(\xi,x,y) := ((\Pi)^{ss})^{-1}(\Pi^{ss}(\xi,x,y))$ and $\eta := \{\eta(\xi,x,y) : (\xi,x,y) \in [0,1]^2 \times \mathbb{R}\}$. Further, let

$$F\eta(\xi,x,y) := F\left(\eta(F^{-1}(\xi,x,y))\right) = \eta(\xi,x,y) \cap \left([0,1] \times R_{k(x)}\right)$$

and $F\eta := \{F\eta(\xi,x,y) : (\xi,x) \in [0,1]^2 \times \mathbb{R}\}$. We define

$$\mu^{F\eta}_{(\xi,z)}(A) := \frac{\mu^{\mathrm{ext}}_{(\xi,z)}\left(R_{k(x)} \cap A\right)}{\mu^{\mathrm{ext}}_{(\xi,z)}\left(R_{k(x)}\right)}$$

for $(\xi,z) \in [0,1]\times\mathbb{R}$ and $A \in \mathcal{B}([0,1]\times\mathbb{R})$. By virtue of the F-invariance of μ^{ext} it is straightforward to verify that both

$$\left(\delta_\xi \otimes \mu^{F\eta}_{\Pi^{ss}(\xi,x,y)}\right)_{(\xi,x,y)\in[0,1]^2\times\mathbb{R}} \qquad \text{and} \qquad \left(\left(\delta_{\rho_{k(x)}(\xi)} \otimes \mu^{\mathrm{ext}}_{\Pi^{ss}\circ F^{-1}(\xi,x,y)}\right) \circ F^{-1}\right)_{(\xi,x,y)\in[0,1]^2\times\mathbb{R}}$$

are conditional probability distributions of μ^{ext} with respect to $F\eta$.

Thus due to their uniqueness we have for ν^{ext}-a.a. (ξ,x) that

$$\begin{aligned}
\mu^{F\eta}_{q(\xi,x)}\left(R_{\omega_0,\omega_1,\ldots,\omega_N}\right) &= \delta_\xi \otimes \mu^{F\eta}_{q(\xi,x)}\left([0,1] \times R_{(\omega_0,\omega_1,\ldots,\omega_N)}\right)\\
&= \left(\delta_{\rho_{k(x)}(\xi)} \otimes \mu^{\mathrm{ext}}_{q\circ B^{-1}(\xi,x)}\right)\left(F^{-1}\left([0,1] \times R_{(\omega_0,\omega_1,\ldots,\omega_N)}\right)\right)\\
&= \left(\delta_{\rho_{k(x)}(\xi)} \otimes \mu^{\mathrm{ext}}_{q\circ B^{-1}(\xi,x)}\right)\left(I_{\omega_0} \times R_{(\omega_1,\ldots,\omega_N)}\right)\\
&= \delta_{\rho_{k(x)}(\xi)}(I_{\omega_0}) \cdot \mu^{\mathrm{ext}}_{q\circ B^{-1}(\xi,x)}\left(R_{(\omega_1,\ldots,\omega_N)}\right)
\end{aligned}$$

holds for all $\omega_0,\ldots,\omega_N \in \{0,\ldots,\ell-1\}$ and $N \in \mathbb{N}$. Inserting $(\omega_0,\ldots,\omega_N) = [x]_{N+1}$ finishes the proof. □

Proposition 3.11. *we have*

$$\lim_{N\to\infty} \frac{\log \nu_{(\xi,q_\xi(x))}\left(I_N(x)\right)}{N} = \lim_{N\to\infty} \frac{\log \nu^{\mathrm{ext}}_{(\xi,q_\xi(x))}\left(I_N(x)\right)}{N} = \int \log \nu^{\mathrm{ext}}_{(\xi,q_\xi(x))}\left(I_1(x)\right)\, d\nu^{\mathrm{ext}}(\xi,x)$$

for ν^{ext}-a.a. (ξ,x).

Proof. The second equality follows from Proposition 3.10 due to Birkhoff ergodic theorem, whereas the first one can be derived by Proposition 3.9. □

In view of Proposition 3.11 it is natural to define the conditional entropy with respect to the the strong stable fibres as

$$h^{ss}_\mu := -\int \log \nu^{\mathrm{ext}}_{(\xi,q_\xi(x))}\left(I_1(x)\right)\, d\nu^{\mathrm{ext}}(\xi,x). \tag{23}$$

3.3. **Dimension and entropy formula.** We shall state and prove several formulas.

The following proposition is a version of the maximal inequality. Note that this does not follow from the classical Besicovitch covering theorem directly since the distance from $q_\xi(x)$ to the centre of $H_r(\xi,x)$ depends not only on ξ but also on x.

Proposition 3.12. *There is a $C_h > 0$ such that*

$$\nu\left\{x \in [0,1] : \sup_{r>0} \frac{1}{\nu\circ q_\xi^{-1}(H_r(\xi,x))} \int_{H_r(\xi,x)} g\, d\left(\nu\circ q_\xi^{-1}\right) > \lambda\right\} \leqslant \frac{C_h}{\lambda} \int g\, d\left(\nu\circ q_\xi^{-1}\right)$$

holds for all $\xi \in [0,1]$, $g \in L^1_{\nu\circ q_\xi^{-1}}$ and $\lambda > 0$, where $H_r(\xi,x) := \pi^{ss}_\xi\left(\Sigma_r(\xi,x)\right)$.

Proof. Let $\xi \in [0,1]$, $g \in L^1_{\nu\circ q_\xi^{-1}}$ and $\lambda > 0$ be fixed. We consider

$$E := \left\{x \in [0,1] : \sup_{r>0} \frac{1}{\nu\circ q_\xi^{-1}(H_r(\xi,x))} \int_{H_r(\xi,x)} g\, d\left(\nu\circ q_\xi^{-1}\right) > \lambda\right\}, \text{ and}$$

$$\mathcal{F}_z := \left\{(x,r) \in q_\xi^{-1}(z) \times (0,\infty) : \frac{1}{\nu\circ q_\xi^{-1}(H_r(\xi,x))} \int_{H_r(\xi,x)} g\, d\left(\nu\circ q_\xi^{-1}\right) > \lambda\right\}$$

for $z \in \mathbb{R}$. As $\mathcal{F}_z \neq \emptyset$ for each $z \in q_\xi(E)$, by the axiom of choice we can parametrise $z \mapsto (x_z, r_z)$ so that $(x_z, r_z) \in \mathcal{F}_z$. Let $H(z) := H_{r_z}(\xi, x_z)$. Since we have $B_{C_s^{-1} r_z}(z) \subseteq H(z) \subseteq B_{C_s r_z}(z)$ for each $z \in q_\xi(E)$ by Proposition 2.4, $(H(z))_{z \in q_\xi(E)}$ is a cover of $q_\xi(E)$, which satisfies the assumptions (a) and (b) of Lemma 2.11 with $M := C_s^2$. Thus by that lemma there is a subcover $(H(z_i))_{i \in J}$ with $J \subseteq \mathbb{N}$ satisfying (1)-(3) of the lemma. Since $q_\xi(E) \subseteq \bigcup_{i \in J} H(z_i)$ by (1), we have by (3)

$$\begin{aligned} \nu(E) \quad &\leqslant \quad \nu\left(\bigcup_{i \in J} q_\xi^{-1} H(z_i)\right) \quad \leqslant \quad \sum_{i \in J} \nu \circ q_\xi^{-1}\left(H(z_i)\right) \\ &\leqslant \quad \frac{1}{\lambda} \sum_{i \in J} \int_{H(z_i)} g \, d\left(\nu \circ q_\xi^{-1}\right) \quad \leqslant \quad \frac{\xi_{1,M}}{\lambda} \int g \, d\left(\nu \circ q_\xi^{-1}\right). \end{aligned}$$

□

Proposition 3.13. *For any $\xi \in [0,1]$ and $A \in \mathcal{B}([0,1] \times \mathbb{R})$ we have that*

$$\lim_{r \to 0} \frac{\mu(\Sigma_r(\xi, x) \cap A)}{\mu(\Sigma_r(\xi, x))} = \lim_{r \to 0} \frac{\mu(B^T_{\xi,r}(q_\xi(x)) \cap A)}{\mu\left(B^T_{\xi,r}(q_\xi(x))\right)}$$

holds for ν-a.a. x.

Proof. Let ξ and A be fixed. Then there exists the Radon–Nikodym derivative

$$\rho_A := \frac{d\left(\mu\left((\pi_\xi^{ss})^{-1}(\,\cdot\,) \cap A\right)\right)}{d\left(\mu \circ (\pi_\xi^{ss})^{-1}\right)}.$$

Note that $\mu \circ (\pi_\xi^{ss})^{-1} = \nu \circ q_\xi^{-1}$. Since

$$\begin{aligned} \lim_{r \to 0} \frac{\mu(B^T_{\xi,r}(q_\xi(x)) \cap A)}{\mu\left(B^T_{\xi,r}(q_\xi(x))\right)} \quad &= \quad \lim_{r \to 0} \frac{1}{\nu \circ q_\xi^{-1}(B_r(q_\xi(x)))} \int_{B_r(q_\xi(x))} \rho_A \, d\left(\nu \circ q_\xi^{-1}\right) \\ &= \quad \rho_A \circ q_\xi(x) \end{aligned}$$

for ν-a.a. x by Lemma 2.12, it suffices to show that the left hand side of the claim also converges to $\rho_A \circ q_\xi$. In order to prove this we need another density theorem. Let $g \in L^1_{\nu \circ q_\xi^{-1}}$. We claim that

$$\lim_{r \to 0} \frac{1}{\nu \circ q_\xi^{-1}(H_r(\xi, x))} \int_{H_r(\xi,x)} g \, d\left(\nu \circ q_\xi^{-1}\right) = g \circ q_\xi$$

holds for ν-a.a. x, where $H_r(\xi, x) := \pi_\xi^{ss}\left(\Sigma_r(\xi, x)\right)$. The proof follows then by taking $g = \rho_A$. The following argument mimics the proof of [13, Lemma 4.1.2]. Since the claim is clearly true for any continuous g, it suffices to show that the set of the functions g which satisfy the above equality is norm closed in $L^1_{\nu \circ q_\xi^{-1}}$. Recall that for an arbitrary $g \in L^1_{\nu \circ q_\xi^{-1}}$ there are continuous functions $(g_k)_k$ such that $g_k \to g$ in L^1. Furthermore, we have then

$$\begin{aligned} &\limsup_{r \to 0} \left| \frac{1}{\nu \circ q_\xi^{-1}(H_r(\xi, x))} \int_{H_r(\xi,x)} g \, d\left(\nu \circ q_\xi^{-1}\right) - g(q_\xi(x)) \right| \\ \leqslant \quad &\sup_{r > 0} \frac{1}{\nu \circ q_\xi^{-1}(H_r(\xi, x))} \int_{H_r(\xi,x)} |g - g_k| \, d\left(\nu \circ q_\xi^{-1}\right) \\ &\quad + \limsup_{r \to 0} \left| \frac{1}{\nu \circ q_\xi^{-1}(H_r(\xi, x))} \int_{H_r(\xi,x)} g_k \, d\left(\nu \circ q_\xi^{-1}\right) - g_k(q_\xi(x)) \right| + |g_k(q_\xi(x)) - g(q_\xi(x))| \\ = \quad &\sup_{r > 0} \frac{1}{\nu \circ q_\xi^{-1}(H_r(\xi, x))} \int_{H_r(\xi,x)} |g - g_k| \, d\left(\nu \circ q_\xi^{-1}\right) + |g_k(q_\xi(x)) - g(q_\xi(x))|. \end{aligned}$$

Thus it follows that

$$\begin{aligned}
&\nu\left\{x\in[0,1]:\limsup_{r\to0}\left|\frac{1}{\nu\circ q_\xi^{-1}(H_r(\xi,x))}\int_{H_r(\xi,x)}g\,d\left(\nu\circ q_\xi^{-1}\right)-g(q_\xi(x))\right|>2\lambda\right\}\\
\leqslant\;&\nu\left\{\sup_{r>0}\frac{1}{\nu\circ q_\xi^{-1}(H_r(\xi,x))}\int_{H_r(\xi,x)}|g-g_k|\,d\left(\nu\circ q_\xi^{-1}\right)>\lambda\right\}+\nu\circ q_\xi^{-1}\left\{|g_k-g|>\lambda\right\}\\
\leqslant\;&\frac{C_h}{\lambda}\|g-g_k\|_{L^1}+\frac{1}{\lambda}\|g_k-g\|_{L^1}\quad\overset{k\to\infty}{\longrightarrow}0
\end{aligned}$$

for each $\lambda>0$ by Proposition 3.12 and Markov's inequality. Thus the claim is proved. □

We introduce the quantity

$$d_{M,r}(\xi,x):=\frac{\mu\left(\Sigma_r(\xi,x)\cap R_M(x)\right)}{\mu\left(\Sigma_r(\xi,x)\right)}$$

for $M\in\mathbb{N}$, $r>0$ and $(\xi,x)\in[0,1]^2$.

Proposition 3.14. *Let $\xi\in[0,1]$ and $M\in\mathbb{N}$. Then we have*

$$\lim_{r\to0}d_{M,r}(\xi,x)=\nu_{(\xi,q_\xi(x))}(I_M(x))$$

for ν-a.a. $x\in[0,1]$.

Proof. From (17) and (18) together with Proposition 3.13 we have for ν-a.a. x that

$$\lim_{r\to0}\frac{\mu(\Sigma_r(\xi,x)\cap R_{(\omega_1,\dots,\omega_M)})}{\mu(\Sigma_r(\xi,x))}=\nu_{(\xi,q_\xi(x))}(I_{(\omega_1,\dots,\omega_M)})$$

for all $\omega_1,\dots,\omega_M\in\{0,\dots,\ell-1\}$. Now, inserting $(\omega_1,\dots,\omega_M)=[x]_M$ finishes the proof. □

Proposition 3.15. *We have*

$$\int\sup_{r>0}|\log d_{M,r}(\xi,x)|\,d\nu^{\mathrm{ext}}(\xi,x)<\infty$$

for each $M\in\mathbb{N}$.

Proof. It suffices to prove $\sum_{n=1}^\infty\nu^{\mathrm{ext}}\left\{(\xi,x)\in[0,1]^2:\inf_{r>0}d_{M,r}(\xi,x)<e^{-n}\right\}<\infty$. Observe that

$$\begin{aligned}
&\nu^{\mathrm{ext}}\left\{(\xi,x)\in[0,1]^2:\inf_{r>0}d_{M,r}(\xi,x)<e^{-n}\right\}\\
=\;&\sum_{\omega_1,\dots,\omega_M\in\{0,\dots,\ell-1\}}\nu^{\mathrm{ext}}\left\{(\xi,x)\in[0,1]\times I_{(\omega_1,\dots,\omega_M)}:\inf_{r>0}\frac{\mu\left(\Sigma_r(\xi,x)\cap R_{(\omega_1,\dots,\omega_M)}\right)}{\mu\left(\Sigma_r(\xi,x)\right)}<e^{-n}\right\}.
\end{aligned}$$

Let $(\omega_1,\dots,\omega_M)$ and n be fixed. Further, let $\xi\in[0,1]$ be any of those for which the first assertion of Proposition 2.7 is true. Then we consider

$$E:=\left\{x\in I_{(\omega_1,\dots,\omega_M)}:\inf_{r>0}\frac{\mu\left(\Sigma_r(\xi,x)\cap R_{(\omega_1,\dots,\omega_M)}\right)}{\mu\left(\Sigma_r(\xi,x)\right)}<e^{-n}\right\},\text{ and}$$

$$\mathcal{F}_y:=\left\{(x,r)\in q_\xi^{-1}(y)\cap I_{(\omega_1,\dots,\omega_M)}\times(0,\infty):\frac{\mu\left(\Sigma_r(\xi,x)\cap R_{(\omega_1,\dots,\omega_M)}\right)}{\mu\left(\Sigma_r(\xi,x)\right)}<e^{-n}\right\}.$$

Now, exactly the same argument as in the proof of Proposition 3.12, where we also named the corresponding objects E and $\mathcal{F}_z$, yields a cover $(H(z_i))_{i\in J}$ of $q_\xi(E)$ with $J\subseteq\mathbb{N}$ satisfying the same properties. By the construction we have $H(z_i)=\pi_\xi^{ss}(\Sigma_{r_i}(\xi,x_i))$ if we define $x_i:=x_{z_i}$ and

$r_i := r_{z_i}$ by using the parametrisation there. Recall that $(x_i, r_i) \in \mathcal{F}_{z_i}$. Since this cover satisfies in particular the multiplicity condition (3) of Lemma 2.11 with the constant $\xi_{1,C_s^2} > 0$, we have

$$\begin{aligned}
\nu_\xi^+(E) &\leqslant \nu_\xi^+\left(q_\xi^{-1}\left(\bigcup_{i\in J} H(z_i)\right) \cap I_{(\omega_1,\dots,\omega_M)}\right) \\
&= \mu_\xi^+\left(\bigcup_{i\in J} \Sigma_{r_i}(\xi, x_i) \cap R_{(\omega_1,\dots,\omega_M)}\right) \\
&\leqslant \sum_{i\in J} \mu_\xi^+\left(\Sigma_{r_i}(\xi, x_i) \cap R_{(\omega_1,\dots,\omega_M)}\right) \\
&\leqslant C_\phi^3 \sum_{i\in J} \mu\left(\Sigma_{r_i}(\xi, x_i) \cap R_{(\omega_1,\dots,\omega_M)}\right) \\
&\leqslant C_\phi^3\, e^{-n} \sum_{i\in J} \mu\left(\Sigma_{r_i}(\xi, x_i)\right) \\
&\leqslant C_\phi^3\, \xi_{1,C_s^2}\, e^{-n}.
\end{aligned}$$

As this is true for all $(\omega_1, \dots, \omega_M) \in \{0, \dots, \ell-1\}^M$ and $n \in \mathbb{N}$, and for ν^--a.a. ξ, by (15) we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} \nu^{\mathrm{ext}}\left\{(\xi, x) \in [0,1]^2 : \inf_{r>0} d_r(\xi, x) < e^{-n}\right\} \\
= &\sum_{\omega_1,\dots,\omega_M \in\{0,\dots,\ell-1\}} \sum_{n=1}^{\infty} \int \nu_\xi^+\left\{x \in I_{(\omega_1,\dots,\omega_M)} : \inf_{r>0} d_r(\xi, x) < e^{-n}\right\} d\nu^-(\xi) \leqslant \frac{C_\phi^3\, \xi_{1,C_s^2}\, e}{e-1}\, \ell^M.
\end{aligned}$$

□

Proposition 3.16. *We have*

$$\begin{aligned}
\mu\left(\Sigma_{\lambda^{MN}(x)}(\xi, x)\right) &= \mu\left(\Sigma_1(B^{-MN}(\xi, x))\right) \cdot \prod_{k=0}^{N-1} \frac{\mu\left(\Sigma_{\lambda^{M(N-k)}(\tau^{Mk}(x))}(B^{-Mk}(\xi, x))\right)}{\mu\left(\Sigma_{\lambda^{M(N-k-1)}(\tau^{M(k+1)}(x))}(B^{-M(k+1)}(\xi, x))\right)} \\
&\in [C_\phi^{-3N}, C_\phi^{3N}] \cdot \mu\left(\Sigma_1(B^{-MN}(\xi, x))\right) \cdot \prod_{k=0}^{N-1} \frac{\nu(I_M(\tau^{Mk}(x)))}{d_{M,\lambda^{M(N-k)}(\tau^{Mk}(x))}(B^{-Mk}(\xi, x))}
\end{aligned}$$

for all $(\xi, x) \in [0,1]^2$ *and* $M, N \in \mathbb{N}$.

Proof. The fist line is only a rewriting. For the second line, observe that we have

$$\begin{aligned}
&\left\{v \in [0,1] : (v, W(v)) \in \Sigma_{\lambda^{M(N-k)}(\tau^{Mk}(x))}(B^{-Mk}(\xi, x)) \cap R_M\left(\tau^{Mk}(x)\right)\right\} \\
= &\left\{v \in I_M(\tau^{Mk}(x)) : \left|\ell^{ss}_{\left(\rho_{[x]_{Mk}}(\xi), v, W(v)\right)}(\tau^{Mk}(x)) - W(\tau^{Mk}(x))\right| \leqslant \lambda^{M(N-k)}(\tau^{Mk}(x))\right\} \\
= &\left\{v \in I_M(\tau^{Mk}(x)) : \left|\ell^{ss}_{\left(B^{-M}\left(\rho_{[x]_{Mk}}(\xi), v\right), W(\tau^M(v))\right)}(\tau^{M(k+1)}(x)) - W(\tau^{M(k+1)}(x))\right|\right. \\
&\qquad\qquad \left.\leqslant \frac{\lambda^{M(N-k)}(\tau^{Mk}(x))}{\lambda^M(\tau^{Mk}(x))}\right\} \\
= &\left\{v \in I_M(\tau^{Mk}(x)) : \left|\ell^{ss}_{\left(\rho_{[x]_{M(k+1)}}(\xi), \tau^M(v), W(\tau^M(v))\right)}(\tau^{M(k+1)}(x)) - W(\tau^{M(k+1)}(x))\right|\right. \\
&\qquad\qquad \left.\leqslant \lambda^{M(N-k-1)}(\tau^{M(k+1)}(x))\right\} \\
= &\; I_M(\tau^{Mk}(x)) \cap \tau^{-M}\left\{v \in [0,1] : (v, W(v)) \in \Sigma_{\lambda^{M(N-k-1)}(\tau^{M(k+1)}(x))}(B^{-M(k+1)}(\xi, x))\right\}
\end{aligned}$$

for $k = 0, \ldots, \ell - 1$ since $[v]_M = [\tau^{Mk} x]_M$. Thus by Proposition 2.6 we have

$$\begin{aligned} & \mu\left(\Sigma_{\lambda^{M(N-k)}(\tau^{Mk}(x))}(B^{-Mk}(\xi, x)) \cap R_M\left(\tau^{Mk}(x)\right)\right) \\ \in \quad & [C_\phi^{-3}, C_\phi^3] \cdot \nu(I_M(\tau^{Mk}(x))) \cdot \mu\left(\Sigma_{\lambda^{M(N-k-1)}(\tau^{M(k+1)}(x))}(B^{-M(k+1)}(\xi, x))\right) \end{aligned}$$

for $k = 0, \ldots, N - 1$. The claim follows by inserting these into the first line. □

Proposition 3.17. *Let $M \in \mathbb{N}$. Then we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \log d_{M,\lambda^{M(N-k)}(\tau^{Mk}(x))}(B^{-Mk}(\xi, x)) = \int \log \nu_{(\xi, q_\xi(x))}(I_M(x))\, d\nu^{\text{ext}}(\xi, x)$$

for ν^{ext}-a.a. (ξ, x).

Proof. Let $g_M(\xi, x) := \log \nu_{(\xi, q_\xi(x))}(I_M(x))$. By Birkhoff ergodic theorem we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} g_M(B^{-Mk}(\xi, x)) = \int g_M\, d\nu^{\text{ext}}$$

for ν^{ext}-a.a. (ξ, x). Furthermore, again by Birkhoff ergodic theorem we have

$$\begin{aligned} & \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{k=0}^{N-1} \log d_{M,\lambda^{M(N-k)}(\tau^{Mk}(x))}(B^{-Mk}(\xi, x)) - \frac{1}{N} \sum_{k=0}^{N-1} g_M(B^{-Mk}(\xi, x)) \right| \\ \leqslant \quad & \limsup_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \left| \log d_{M,\lambda^{M(N-k)}(\tau^{Mk}(x))}(B^{-Mk}(\xi, x)) - g_M(B^{-Mk}(\xi, x)) \right| \\ \leqslant \quad & \limsup_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} D_R(B^{-Mk}(\xi, x)) \quad = \int D_R\, d\nu^{\text{ext}} \end{aligned}$$

for ν^{ext}-a.a. (ξ, x), where $D_R(\xi, x) := \sup_{r\in(0,R)} |\log d_{M,r}(\xi, x) - g_M(\xi, x)|$. Letting $R \to 0$ finishes the proof since the last integral goes to zero by Propositions 3.14 and 3.15 due to the dominated convergence theorem. □

Proposition 3.18. *We have*

$$\begin{aligned} & \limsup_{r\to 0} \left| \frac{\log \mu(\Sigma_r(\xi, x))}{\log r} - \frac{\int \log \nu_{(\xi, q_\xi(x))}(I_M(x))\, d\nu^{\text{ext}}(\xi, x) - \int \log \nu(I_M(x))\, d\nu(x)}{-M \int \log \lambda\, d\nu} \right| \\ \leqslant \quad & \frac{3 \log C_\phi}{-M \int \log \lambda\, d\nu} \end{aligned}$$

for ν^{ext}-a.a. (ξ, x).

Proof. Similarly to Proposition 2.9, one can show

$$\limsup_{r\to 0} \left| \frac{\log \mu(\Sigma_r(\xi, x))}{\log r} - a \right| = \limsup_{N\to\infty} \left| \frac{\log \mu(\Sigma_{\lambda^{MN}(x)}(\xi, x))}{\log \lambda^{MN}(x)} - a \right|$$

for any $(\xi, x) \in [0, 1]^2$ and $a \in \mathbb{R}$. Furthermore, we have $\log \mu(\Sigma_1) \in L^1_{\nu^{\text{ext}}}$ since $\inf_{(\xi,x)} \mu(\Sigma_1(\xi, x)) > 0$ by Proposition 2.3 and the Gibbs property. Thus the claim follows from Proposition 3.16 since we have

$$\begin{aligned} & \lim_{N\to\infty} \frac{\log\left(\prod_{k=0}^{N-1} \frac{\nu(I_M(\tau^{Mk}(x)))}{d_{M,\lambda^{M(N-k)}(\tau^{Mk}(x))}(B^{-Mk}(\xi, x))} \right)}{\log \lambda^{MN}(x)} \\ & \quad = \frac{\int \log \nu_{(\xi, q_\xi(x))}(I_M(x))\, d\nu^{\text{ext}}(\xi, x) - \int \log \nu(I_M(x))\, d\nu(x)}{-M \int \log \lambda\, d\nu} \end{aligned}$$

by Proposition 3.17 and Birkhoff ergodic theorem. □

Recall that h_ν and h_μ^{ss} are defined in (8) and (23), respectively.

Lemma 3.19. *We have*

$$\lim_{r\to 0}\frac{\log\mu(B^T_{\xi,r}(q_\xi(x)))}{\log r}=\lim_{r\to 0}\frac{\log\mu(\Sigma_r(\xi,x))}{\log r}=\frac{h_\nu-h^{ss}_\mu}{-\int\log\lambda\,d\nu}$$

for ν^{ext}-a.a. (ξ,x).

In particular, $\nu\circ q_\xi^{-1}$ is exact dimensional for ν^--a.a. ξ with

$$\dim_H(\nu\circ q_\xi^{-1})=\frac{h_\nu-h^{ss}_\mu}{-\int\log\lambda\,d\nu}.$$

Proof. The second equality in the first claim follows from Proposition 3.18 by letting $M\to\infty$ since we have

$$\begin{aligned}-M\cdot h^{ss}_\mu &= M\int\log\nu^{\mathrm{ext}}_{(\xi,q_\xi(x))}(I_1(x))\,d\nu^{\mathrm{ext}}(\xi,x)\\ &= \int\log\nu^{\mathrm{ext}}_{(\xi,q_\xi(x))}(I_M(x))\,d\nu^{\mathrm{ext}}(\xi,x)\\ &\in [\log C_\phi^{-6},\log C_\phi^{6}]+\int\log\nu_{(\xi,q_\xi(x))}(I_M(x))\,d\nu^{\mathrm{ext}}(\xi,x)\end{aligned}$$

by Propositions 3.9 and 3.10. On the other hand, the first equality is true in view of $\Sigma_{C_s^{-1}r}(\xi,x)\subseteq B^T_{\xi,r}(q_\xi(x))\subseteq\Sigma_{C_sr}(\xi,x)$, where $C_s>0$ denotes the constant of Proposition 2.4.

Now, we prove the remaining claim on the dimension. Let $\gamma_2:=-(h_\nu-h^{ss}_\mu)/\int\log\lambda\,d\nu$. As $\mu(B^T_{\xi,r}(y))=\nu\circ q_\xi^{-1}(B_r(y))$, we have by (15)

$$\begin{aligned}&\int\nu^+_\xi\circ q_\xi^{-1}\left\{y\in\mathbb{R}:\lim_{r\to 0}\frac{\log\nu\circ q_\xi^{-1}(B_r(y))}{\log r}=\gamma_2\right\}d\nu^-(\xi)\\ =\;&\nu^{\mathrm{ext}}\left\{(\xi,x)\in[0,1]^2:\lim_{r\to 0}\frac{\log\nu\circ q_\xi^{-1}(B_r(q_\xi(x)))}{\log r}=\gamma_2\right\}=1.\end{aligned}$$

Thus by Proposition 2.7 we can conclude

$$\nu\circ q_\xi^{-1}\left\{y\in\mathbb{R}:\lim_{r\to 0}\frac{\log\nu\circ q_\xi^{-1}(B_r(y))}{\log r}=\gamma_2\right\}=1$$

for ν^--a.a. ξ. □

Lemma 3.20. *We have*

$$\lim_{r\to 0}\frac{\log\nu_{(x,q_\xi(x))}(B_r(x))}{\log r}=\frac{h^{ss}_\mu}{\int\log\tau'\,d\nu}$$

for ν^{ext}-a.a. (ξ,x).

In particular, $\nu_{(x,q_\xi(x))}$ and $\mu_{(x,q_\xi(x))}$ are exact dimensional for ν^{ext}-a.a. (ξ,x) with

$$\dim_H(\nu_{(x,q_\xi(x))})=\dim_H(\mu_{(x,q_\xi(x))})=\frac{h^{ss}_\mu}{\int\log\tau'\,d\nu}.$$

Proof. By Propositions 2.2 and 3.11 we have

$$\lim_{N\to\infty}\frac{\log\nu_{(x,q_\xi(x))}(I_N(x))}{\log|I_N(x)|}=\frac{h^{ss}_\mu}{\int\log\tau'\,d\nu}$$

for ν^{ext}-a.a. (ξ,x). Therefore, for the first claim it suffices to show that

$$d_{\nu_{(x,q_\xi(x))}}(x)=\frac{\log\nu_{(x,q_\xi(x))}(I_N(x))}{\log|I_N(x)|}$$

holds for ν^{ext}-a.a. (ξ,x). Let $t\in(0,1)$ be arbitrary. By Proposition 2.8 for ν-a.a. x there is a N_x such that

$$B_{|I_N(x)|t^N}(x)\subseteq I_N(x)\subseteq B_{|I_N(x)|}(x)$$

holds for all $N \geqslant N_x$. From this follows for such x and any $\xi \in [0,1]$

$$\begin{aligned} \overline{d}_{\nu_{(x,q_\xi(x))}}(x) &= \limsup_{N\to\infty} \frac{\log \nu_{(x,q_\xi(x))}(B_{|I_N(x)|}(x))}{\log |I_N(x)|} \\ &\leqslant \lim_{N\to\infty} \frac{\log \nu_{(x,q_\xi(x))}(I_N(x))}{\log |I_N(x)|} \\ &\leqslant \liminf_{N\to\infty} \frac{\log \nu_{(x,q_\xi(x))}(B_{|I_N(x)|t^N}(x))}{\log(|I_N(x)|t^N)} \left(1 + \frac{\log t}{\log \min_i |I_i|}\right) \\ &= \underline{d}_{\nu_{(x,q_\xi(x))}}(x) \left(1 + \frac{\log t}{\log \min_i |I_i|}\right) \end{aligned}$$

by virtue of Proposition 2.9, as $\lim_{N\to\infty} \frac{\log |I_{N+1}(x)|}{\log |I_N(x)|} = \lim_{N\to\infty} \frac{\log(|I_{N+1}(x)|t^{N+1})}{\log(|I_N(x)|t^N)} = 1$. Letting $t \nearrow 1$ finishes this proof.

Next, we consider the dimension of $\nu_{(x,q_\xi(x))}$. Let $\gamma_1 := h_\mu^{ss} / \int \log \tau' \, d\nu$. By Lemma 3.7 we have

$$\begin{aligned} &\int \nu^{\text{ext}}_{(\xi,q_\xi(x))} \left\{ \tilde{x} \in [0,1] : \lim_{r\to 0} \frac{\log \nu_{(\xi,q_\xi(x))}(B_r(\tilde{x}))}{\log r} = \gamma_1 \right\} d\nu^{\text{ext}}(\xi, x) \\ = \;& \nu^{\text{ext}} \left\{ (\xi, x) \in [0,1]^2 : \lim_{r\to 0} \frac{\log \nu_{(\xi,q_\xi(x))}(B_r(x))}{\log r} = \gamma_1 \right\} \quad = 1. \end{aligned}$$

Thus by Proposition 3.9 we can conclude

$$\nu_{(\xi,q_\xi(x))} \left\{ \tilde{x} \in [0,1] : \lim_{r\to 0} \frac{\log \nu_{(\xi,q_\xi(x))}(B_r(\tilde{x}))}{\log r} = \gamma_1 \right\} = 1$$

for ν^{ext}-a.a. (ξ, x).

Finally, the pointwise dimension of $\mu_{(x,q_\xi(x))}$ at $(v, \ell^{ss}_{(\xi,x,W(x))}(v))$ coincides with that of $\nu_{(x,q_\xi(x))}$ at v for all $v \in [0,1]$ and $(\xi, x) \in [0,1]^2$ since the strong stable fibres are bi-Lipschitz continuous. □

The next lemma describes the product structure of μ from the dimension theoretic point of view. Its proof mimics that of [4, Proposition 3.7] partially.

Lemma 3.21. *μ is exact dimensional with*

$$\dim_H(\mu) = \dim_H(\mu_{(\xi,q_\xi(x))}) + \dim_H(\nu \circ q_\xi^{-1})$$

for ν^{ext}-a.a. (ξ, x).

Proof. Let $\beta := \exp(\int \log \lambda \, d\nu)$, $\gamma_1 := h_\mu^{ss} / \int \log \tau' \, d\nu$ and $\gamma_2 := -(h_\mu^+ - h_\mu^{ss}) / \int \log \lambda \, d\nu$. Further, let $C_0 := \|X_3\|_\infty + 1$ so that

$$B_r(x, W(x)) \subseteq \{(v, y) \in \Sigma_{C_0 r}(\xi, x) : |x - v| \leqslant r\} \tag{24}$$

holds for all $x \in [0,1]$ and $r > 0$. We shall prove that the lower and upper pointwise dimension are almost surely bounded by $\gamma_1 + \gamma_2$ from below and above, respectively.

For the lower bound, let $\varepsilon > 0$ be arbitrary. For $N_0 \in \mathbb{N}$ let $\mathcal{G}_{N_0}$ denote the set of those $(\xi, x) \in [0,1]^2$ which satisfy

$$\nu_{(x,q_\xi(x))}(B_{2\beta^N}(x)) \in [\beta^{N(\gamma_1+\varepsilon)}, \beta^{N(\gamma_1-\varepsilon)}], \text{ and} \tag{25}$$

$$\mu(\Sigma_{C_0\beta^N}(\xi, x)) \in [\beta^{N(\gamma_2+\varepsilon)}, \beta^{N(\gamma_2-\varepsilon)}] \tag{26}$$

for all $N \geqslant N_0$. Then we have $\lim_{N\to\infty} \int \nu_\xi^+((\mathcal{G}_N)_\xi) \, d\nu^-(\xi) = \lim_{N\to\infty} \nu^{\text{ext}}(\mathcal{G}_N) = 1$ by (15), Lemmas 3.19 and 3.20, where $(\mathcal{G}_N)_\xi := \{x \in [0,1] : (\xi, x) \in \mathcal{G}_N\}$. By the monotone convergence theorem and Proposition 2.7, we have thus $\lim_{N\to\infty} \nu((\mathcal{G}_N)_\xi) = 1$ for ν^--a.a. ξ.

Let such a $\xi \in [0,1]$ be fixed. Then we choose a $N_0 \in \mathbb{N}$ so large that $\nu((\mathcal{G}_{N_0})_\xi) \geqslant 1 - \varepsilon$. By Lemma 2.12 and Egorov's theorem there are $N_1 \geqslant N_0$ and $\mathcal{G}' \subseteq (\mathcal{G}_{N_0})_\xi$ with $\nu(\mathcal{G}') > 1 - 2\varepsilon$ such that

$$\mu\left(B_{\beta^N}(x, W(x)) \cap (\mathcal{G}_{N_0})_\xi \times \mathbb{R}\right) \geqslant \frac{1}{2} \mu\left(B_{\beta^N}(x, W(x))\right) \tag{27}$$

holds for all $x \in \mathcal{G}'$ and $N \geqslant N_1$.

We claim that

$$\mu_{(\xi,z)}\left(B_{\beta^N}(x,W(x))\cap(\mathcal{G}_{N_0})_\xi\times\mathbb{R}\right)\leqslant\beta^{N(\gamma_2-\varepsilon)} \tag{28}$$

holds for all $z\in\mathbb{R}$, $x\in\mathcal{G}'$ and $N\geqslant N_1$. If $\mu_{(\xi,z)}\left(B_{\beta^N}(x,W(x))\cap(\mathcal{G}_{N_0})_\xi\times\mathbb{R}\right)=0$, the claim is trivial. Thus we assume that there is a $(\tilde{x},\ell^{ss}_{(\xi,0,z)}(\tilde{x}))\in B_{\beta^N}(x,W(x))\cap(\mathcal{G}_{N_0})_\xi\times\mathbb{R}$. In this case, $\mu_{(\xi,z)}\left(B_{\beta^N}(x,W(x))\cap(\mathcal{G}_{N_0})_\xi\times\mathbb{R}\right)\leqslant\nu_{(\xi,z)}(B_{2\beta^N}(\tilde{x}))=\nu_{(\xi,q_\xi(\tilde{x}))}(B_{2\beta^N}(\tilde{x}))\leqslant\beta^{N(\gamma_2-\varepsilon)}$ by (25).

Now, from (24), (26), (27) and (28) together with Lemma 3.4 follows

$$\begin{aligned}\mu\left(B_{\beta^N}(x,W(x))\right) &\leqslant 2\mu\left(B_{\beta^N}(x,W(x))\cap(\mathcal{G}_{N_0})_\xi\times\mathbb{R}\right)\\ &\leqslant 2\int_{\pi^{ss}_\xi(\Sigma_{C_0\beta^N}(\xi,x))}\mu_{(\xi,z)}\left(B_{\beta^N}(x,W(x))\cap(\mathcal{G}_{N_0})_\xi\times\mathbb{R}\right)\,d(\nu\circ q_\xi^{-1})(z)\\ &\leqslant 2\beta^{N(\gamma_1+\gamma_2-2\varepsilon)}\end{aligned}$$

for all $x\in\mathcal{G}'$ and $N\geqslant N_1$. Therefore by Proposition 2.9 we have

$$\liminf_{r\to0}\frac{\log\mu\left(B_r(x,W(x))\right)}{\log r}\geqslant\gamma_1+\gamma_2-2\varepsilon$$

for $x\in\mathcal{G}'$. As $\nu(\mathcal{G}')>1-2\varepsilon$ and $\varepsilon>0$ was arbitrary, the lower estimate is proved ν-a.s..

Now, we prove the upper bound. Again, let $\varepsilon>0$ be arbitrary. For $(\xi,x)\in[0,1]^2$ and $N_2\in\mathbb{N}$ let $\mathcal{F}_{(\xi,x),N_2}$ denote the set of those $v\in[0,1]$ which satisfy

$$\begin{aligned} W(v) &= \ell^{ss}_{(\xi,x,q_\xi(x))}(v), &(29)\\ \lambda^N(v) &\in [\beta^{N(1+\varepsilon)},\beta^{N(1-\varepsilon)}], &(30)\\ \nu_{(x,q_\xi(x))}(B_{\beta^N}(v)) &\in [\beta^{N(\gamma_1+\varepsilon)},\beta^{N(\gamma_1-\varepsilon)}], &(31)\\ \nu(I_N(v)) &\in [e^{-N(h_\nu+\varepsilon)},e^{-N(h_\nu-\varepsilon)}], &(32)\\ \nu_{(\xi,q_\xi(x))}(I_N(v)) &\in [e^{-N(h^{ss}_\mu+\varepsilon)},e^{-N(h^{ss}_\mu-\varepsilon)}],\text{ and} &(33)\\ |I_N(v)| &\in [0,\beta^N] &(34)\end{aligned}$$

for all $N\geqslant N_2$. Then we have $\lim_{N\to\infty}\int\nu_{(\xi,q_\xi(x))}(\mathcal{F}_{(\xi,x),N})\,d\nu(\xi,x)=1$ by Lemma 3.4 in view of (19), Birkhoff ergodic theorem, Lemma 3.20, (9), Proposition 3.11, Proposition 2.2 and the fact that $\exp(-\int\log\tau'\,d\nu)<\beta$. By the monotone convergence theorem and Proposition 3.9, we have therefore $\lim_{N\to\infty}\nu_{(\xi,q_\xi(x))}(\mathcal{F}_{(\xi,x),N})=1$ for ν^{ext}-a.a. (ξ,x).

Let such a (ξ,x) be fixed and let N_2 be so large that $\nu_{(\xi,q_\xi(x))}(\mathcal{F}_{(\xi,x),N_2})>1-\varepsilon$. By Lemma 2.12 and Egorov's theorem, there are $N_3\geqslant N_2$ and $\mathcal{F}'\subseteq\mathcal{F}_{(\xi,x),N_2}$ with $\nu_{(\xi,q_\xi(x))}(\mathcal{F}')>1-2\varepsilon$ such that

$$\nu_{(\xi,q_\xi(x))}\left(B_{\beta^N}(v)\cap\mathcal{F}_{(\xi,x),N_2}\right)\geqslant\frac{1}{2}\nu_{(\xi,q_\xi(x))}\left(B_{\beta^N}(v)\right) \tag{35}$$

holds for all $v\in\mathcal{F}'$ and $N\geqslant N_3$. Let a $N\geqslant N_3$ be fixed. Then we choose $(u_i)_{i=0}^{\ell^N-1}$ so that $I_N(u_i)\neq I_N(u_j)$ for $i\neq j$ and that $u_i\in\mathcal{F}_{(\xi,x),N_2}$ whenever $I_N(u_i)\cap\mathcal{F}_{(\xi,x),N_2}\neq\emptyset$. By (31), (35) and (33) we obtain

$$\begin{aligned}\beta^{N(\gamma_1+\varepsilon)} \leqslant \nu_{(\xi,q_\xi(x))}\left(B_{\beta^N}(v)\right) &\leqslant 2\,\nu_{(\xi,q_\xi(x))}\left(B_{\beta^N}(v)\cap\mathcal{F}_{(\xi,x),N_2}\right)\\ &\leqslant 2\sum_{i\in A_v}\nu_{(\xi,q_\xi(x))}(I_N(u_i)) \leqslant \sharp A_v\cdot e^{-N(h^{ss}_\mu-\varepsilon)}\end{aligned} \tag{36}$$

for all $v\in\mathcal{F}'$, where $A_v:=\{i:I_N(u_i)\cap B_{\beta^N}(v)\cap\mathcal{F}_{(\xi,x),N_2}\neq\emptyset\}$ and $\sharp A_v$ denotes the cardinality of A_v. On the other hand, by Proposition 2.3 and (30) we have

$$\{(w,W(w)):w\in I_N(u_i)\}\subseteq I_N(u_i)\times B_{C_m\lambda^N(u_i)}(W(u_i))\subseteq I_N(u_i)\times B_{C_m\beta^{N(1-\varepsilon)}}(W(u_i))$$

Furthermore, $|W(v)-W(u_i)|=\left|\ell^{ss}_{(\xi,x,W(x))}(v)-\ell^{ss}_{(\xi,x,W(x))}(u_i)\right|\leqslant C_0\beta^N$ holds for all $v\in\mathcal{F}'$ and $i\in A_v$ by (29) since $|v-u_i|\leqslant\beta^N$ by the definition of A_v. Thus we have

$$\{(w,W(w)):w\in I_N(u_i)\}\subseteq B_{\tilde{C}\,\beta^{N(1-\varepsilon)}}(v,W(v)) \tag{37}$$

for all $v \in \mathcal{F}'$ and $i \in A_v$, where $\tilde{C} := 1 + C_m + C_0$. Now, from (32), (36) and (37) follows

$$\mu\left(B_{\tilde{C}\,\beta^{N(1-\varepsilon)}}(v, W(v))\right) \geqslant \sum_{i \in A_v} \nu(I_N(u_i)) \geqslant \sharp A_v \cdot e^{-N(h_\nu+\varepsilon)} \geqslant \beta^{N(\gamma_1+\varepsilon)} \cdot e^{-N(h_\nu - h_\mu^{ss} + 2\varepsilon)}$$

for all $v \in \mathcal{F}'$.

As the above consequence is true for all $v \in \mathcal{F}'$ and $N \geqslant N_3$, by Proposition 2.9 we have

$$\begin{aligned}\limsup_{r\to 0} \frac{\log \mu(B_r(v, W(v)))}{\log r} &= \limsup_{N\to\infty} \frac{\log \mu\left(B_{\tilde{C}\,\beta^{N(1-\varepsilon)}}(v, W(v))\right)}{\log \beta^{N(1-\varepsilon)}} \\ &\leqslant \left(\gamma_1 + \varepsilon + \frac{h_\mu^{ss} - h_\nu - 2\varepsilon}{-\log\beta}\right)\frac{1}{1-\varepsilon}\end{aligned}$$

for all $v \in \mathcal{F}'$. As $\nu_{(\xi, q_\xi(x))}(\mathcal{F}') \geqslant 1 - 2\varepsilon$ and $\varepsilon > 0$ was arbitrary, letting $\varepsilon \to 0$ yields in view of Lemma 3.19 that

$$\limsup_{r\to 0} \frac{\log \mu\left(B_r(v, W(v))\right)}{\log r} \leqslant \gamma_1 + \gamma_2$$

holds for $\nu_{(\xi, q_\xi(x))}$-a.a. v. Finally, since this is true for ν^{ext}-a.a. (ξ, x), by Lemma 3.4 we can conclude

$$\begin{aligned}&\nu\left\{x \in [0,1] : \limsup_{r\to 0} \frac{\log \mu\left(B_r(x, W(x))\right)}{\log r} \leqslant \gamma_1 + \gamma_2\right\} \\ = &\int \nu_{(\xi, q_\xi(x))}\left\{v \in [0,1] : \limsup_{r\to 0} \frac{\log \mu\left(B_r(v, W(v))\right)}{\log r} \leqslant \gamma_1 + \gamma_2\right\} d\nu(x) \quad = 1.\end{aligned}$$

□

Proof of Theorem 1. This follows from Lemmas 3.19, 3.20 and 3.21. □

4. Ledrappier's lemma

In this section, let $\nu \in \mathcal{P}([0,1])$ be an invariant measure, that is not necessarily a Gibbs measure. Instead we suppose the dimension theoretic product structure of μ which is always satisfied in case of Gibbs measures as Lemma 3.21 together with Proposition 2.7 shows. More precisely, suppose that $\nu_{(\xi, q_\xi(x))}$ and $\nu \circ q_\xi^{-1}$ are for $\nu^- \otimes \nu$-a.a. (ξ, x) exact dimensional with constant dimensions, say,

$$\gamma_1 := \dim_H(\nu_{(\xi, q_\xi(x))}) \quad \text{and} \quad \gamma_2 := \dim_H(\nu \circ q_\xi^{-1}), \tag{38}$$

and that μ is also exact dimensional with

$$\dim_H(\mu) = \gamma_1 + \gamma_2. \tag{39}$$

Recall that $\Theta : [0,1]^2 \to \mathbb{R}$ is defined in 11. Now the lemma says the following.

Lemma 4.1 (Ledrappier's Lemma)**.** *Under the above assumption we have*

$$\begin{cases}\dim_H(\nu \circ q_\xi^{-1}) = 1 & \text{if } \dim_H(\mu) \geqslant 1 \\ \dim_H(\nu_{(\xi, q_\xi(x))}) = 0 & \text{if } \dim_H(\mu) < 1\end{cases},$$

whenever the distribution of $\Theta(\,\cdot\,, x)$ *under* ν^- *has Hausdorff dimension* 1 *for* ν*-a.e.* $x \in [0,1]$.

4.1. **Outline of the proof of Lemma 4.1.** As the procedure of the proof is a bit tricky, we sketch our approach fist. We use the following convention.

- Let $B_r(x, y) := B_r((x, y))$.
- Let $\|\cdot\|$ denote the euclidean norm of $\mathbb{R}^2$.
- Let $A_\xi := \{x \in [0,1] : (\xi, x) \in A\}$ for $\xi \in [0,1]$ and $A \subseteq [0,1]^2$.
- Let $\mathbb{P}_x \in \mathcal{P}(\mathbb{R})$ denote the distribution of $\Theta(\,\cdot\,, x)$ under ν^-.
- Let $\alpha \in (0,1)$ be the minimum of the Hölder exponents of g', λ' and $\log\gamma$.
- Let $a := \min\{1, \gamma_1 + \gamma_2\}$.

- Let $|\cdot|_\alpha$ and $|\cdot|_{\mathrm{pw},\alpha}$ denote the α-Hölder and the peacewise α-Hölder seminorm, respectively, i.e. for $\varphi : [0,1] \to \mathbb{R}$ let
$$|\varphi|_\alpha := \sup_{x,\tilde{x}\in(0,1)} \frac{|\varphi(x)-\varphi(\tilde{x})|}{|x-\tilde{x}|^\alpha} \quad \text{and} \quad |\varphi|_{\mathrm{pw},\alpha} := \max_{i\in\{0,\dots,\ell-1\}} \sup_{x,\tilde{x}\in I_i^\circ} \frac{|\varphi(x)-\varphi(\tilde{x})|}{|x-\tilde{x}|^\alpha}.$$

Suppose that we have $\dim_H(\mathbb{P}_x) = 1$ for ν-a.e. $x \in [0,1]$.

Let $\delta \in (0, \gamma_1+\gamma_2)$ and $t \in (\frac{1}{1+\alpha}, 1)$ be fixed. In view of (39) and the above assumption on $\mathbb{P}_x$ together with Fubini's theorem, we can choose an $E > 0$ so large that the set
$$G := \left\{(\xi,x) \in [0,1]^2 : \begin{array}{l} \mathbb{P}_x\left(B_\eta\left(\Theta(\xi,x)\right)\right) \leqslant E\,\eta^{1-2\delta} \text{ and} \\ \mu\left(B_\eta(x,W(x))\right) \leqslant E\,\eta^{\gamma_1+\gamma_2-a\delta} \end{array} \text{ for } \forall\eta > 0\right\}$$
has a positive $\nu^- \otimes \nu$ measure. Then we consider
$$\begin{aligned} b(\xi,x,r,t) &:= \mu\left(G_\xi \times \mathbb{R} \cap \Sigma_r(\xi,x) \cap B_{r^t}(x,W(x))\right) \\ &= \nu\left\{v \in G_\xi : \left|\ell^{ss}_{(\xi,v,W(v))}(x) - W(x)\right| \leqslant r \text{ and } \|(v,W(v)) - (x,W(x))\| \leqslant r^t\right\}. \end{aligned}$$
In the following subsections, as (43) and (46), we shall prove that
$$a(1-2\delta) + t(\gamma_1+\gamma_2-a) \leqslant \limsup_{r\to 0} \frac{\log b(\xi,x,r,t)}{\log r} \leqslant t\gamma_1 + \gamma_2 + 3\delta \tag{40}$$
holds on some set of (ξ,x) with positive $\nu^- \otimes \nu$ measure.

Once this has been proved, we are able to conclude the proof as follows.

Proof of Lemma 4.1. Assume first that $\gamma_1+\gamma_2 \geqslant 1$. Then the inequality (40) implies
$$1 + t(\gamma_1+\gamma_2-1) \leqslant t\gamma_1 + \gamma_2$$
for a $t \in (\frac{1}{1+\alpha}, 1)$ since $\delta \in (0,\gamma_1+\gamma_2)$ was arbitrary. This can be rewritten as $1-\gamma_2 \leqslant t(1-\gamma_2)$. The last expression is only possible in case $\gamma_2 = 1$ since $\gamma_2 = \dim_H(\mu\circ q_\xi^{-1}) \in [0,1]$.

Now, assume that $\gamma_1+\gamma_2 < 1$. In this case the inequality (40) implies
$$\gamma_1+\gamma_2 \leqslant t\gamma_1+\gamma_2,$$
for a $t \in (\frac{1}{1+\alpha}, 1)$, or equivalently, $\gamma_1 \leqslant t\gamma_1$. Thus $\gamma_1 = 0$. □

4.2. **Lower estimate.** The goal of this subsection is to prove the lower estimate of (40).

Let $x \in [0,1]$. By Fubini's theorem we have
$$\begin{aligned} &\int_{\{\xi\in[0,1]:(\xi,x)\in G\}} b(\xi,x,r,t)\,d\nu^-(\xi) \\ =\ &\int_{\{(v,W(v))\in B_{r^t}(x,W(x))\}} \nu^-\left\{\xi \in [0,1] : v,x \in G_\xi \text{ and } \left|\ell^{ss}_{(\xi,v,W(v))}(x) - W(x)\right| \leqslant r\right\} d\nu(v). \end{aligned}$$
Let $L^\theta_{(x,y)} : [0,1] \to \mathbb{R}$ be the linear function through (x,y) with slope θ, i.e.
$$L^\theta_{(x,y)}(v) := \theta(v-x) + y.$$
We show now several properties of this projection function.

Proposition 4.2. *Let $Y > 0$. Then there is a $C_Y > 0$ such that*
$$\left|L^{X_3(\xi,x,y)}_{(x,y)}(v) - \ell^{ss}_{(\xi,x,y)}(v)\right| \leqslant C_Y|x-v|^{1+\alpha}$$
for all $(\xi,x) \in [0,1]^2$, $v \in [0,1]$ and $y \in [-Y,Y]$.

Proof. Since $\left(L^{X_3(\xi,x,y)}_{(x,y)}\right)'(v) = X_3(\xi,x,y)$ and $\left(\ell^{ss}_{(\xi,x,y)}\right)'(v) = X_3\left(\xi, v, \ell^{ss}_{(\xi,x,y)}(v)\right)$, it suffices to show
$$\sup_{(\xi,x,y)\in[0,1]^2\times\mathbb{R}} \left|v \mapsto X_3\left(\xi,v,\ell^{ss}_{(\xi,x,y)}(v)\right)\right|_\alpha < \infty.$$

Recall that by (10) we have

$$\begin{aligned} & X_3\left(\xi, v, \ell^{ss}_{(\xi,x,y)}(v)\right) \qquad (41)\\ = \quad & -\sum_{n=1}^{\infty}\gamma^n\left(\rho_{[\xi]_n}(v)\right)\cdot F^{n-1}_{(\xi,x)}\left(\ell^{ss}_{(\xi,x,y)}(v)\right)\cdot\lambda'\left(\rho_{[\xi]_n}(v)\right)-\sum_{n=1}^{\infty}\gamma^n\left(\rho_{[\xi]_n}(v)\right)\cdot g'\left(\rho_{[\xi]_n}(v)\right).\end{aligned}$$

First, we consider the latter sum in the above expression. Since there is a $C_1>0$ such that

$$\begin{aligned}\log\gamma^n\left(\rho_{[\xi]_n}(v)\right)-\log\gamma^n\left(\rho_{[\xi]_n}(\tilde v)\right) &= \sum_{j=1}^{n}\log\gamma\left(\rho_{[\xi]_j}(v)\right)-\log\gamma\left(\rho_{[\xi]_j}(\tilde v)\right)\\ &\leqslant |\log\gamma|_{\mathrm{pw},\alpha}\sum_{j=1}^{n}|\rho_{[\xi]_j}(v)-\rho_{[\xi]_j}(\tilde v)|^{\alpha}\\ &\leqslant C_1\,|v-\tilde v|^{\alpha}\end{aligned}$$

holds for all $\xi, v, \tilde v\in[0,1]$ and $n\in\mathbb{N}$, we have

$$\begin{aligned}\gamma^n\left(\rho_{[\xi]_n}(v)\right)-\gamma^n\left(\rho_{[\xi]_n}(\tilde v)\right) &= \gamma^n\left(\rho_{[\xi]_n}(v)\right)\left(1-\frac{\gamma^n\left(\rho_{[\xi]_n}(\tilde v)\right)}{\gamma^n\left(\rho_{[\xi]_n}(v)\right)}\right)\\ &\leqslant \gamma^n\left(\rho_{[\xi]_n}(v)\right)\left(e^{\left|\log\gamma^n\left(\rho_{[\xi]_n}(v)\right)-\log\gamma^n\left(\rho_{[\xi]_n}(\tilde v)\right)\right|}-1\right)\\ &\leqslant C_2\,\|\gamma\|_{\infty}^{n}\,|v-\tilde v|^{\alpha} \qquad (42)\end{aligned}$$

for $C_2:=(e^{C_1}-1)C_1^{-1}$, where we used the inequality $e^x-1\leqslant(e^M-1)M^{-1}x$ for $0\leqslant x\leqslant M$. Further, from

$$\begin{aligned} & \gamma^n\left(\rho_{[\xi]_n}(v)\right)\,g'\left(\rho_{[\xi]_n}(v)\right)-\gamma^n\left(\rho_{[\xi]_n}(\tilde v)\right)\,g'\left(\rho_{[\xi]_n}(\tilde v)\right)\\ \leqslant\quad & \gamma^n\left(\rho_{[\xi]_n}(v)\right)\left(g'\left(\rho_{[\xi]_n}(v)\right)-g'\left(\rho_{[\xi]_n}(\tilde v)\right)\right)+\left(\gamma^n\left(\rho_{[\xi]_n}(v)\right)-\gamma^n\left(\rho_{[\xi]_n}(\tilde v)\right)\right)g'\left(\rho_{[\xi]_n}(\tilde v)\right)\\ \leqslant\quad & \left(|g'|_{\mathrm{pw},\alpha}+C_2\right)\|\gamma\|_{\infty}^{n}\,|v-\tilde v|^{\alpha}\end{aligned}$$

follows

$$\left|v\mapsto\sum_{n=1}^{\infty}\gamma^n\left(\rho_{[\xi]_n}(v)\right)\cdot g'\left(\rho_{[\xi]_n}(v)\right)\right|_{\alpha}\leqslant\frac{|g'|_{\mathrm{pw},\alpha}+C_2}{1-\|\gamma\|_{\infty}}.$$

Now, we estimate the first sum of (41). Observe that by (7) we can reformulate it as

$$\begin{aligned} & \sum_{n=1}^{\infty}\gamma^n\left(\rho_{[\xi]_n}(v)\right)\cdot F^{n-1}_{(\xi,x)}\left(\ell^{ss}_{(\xi,x,y)}(v)\right)\cdot\lambda'\left(\rho_{[\xi]_n}(v)\right)\\ = \quad & \ell^{ss}_{(\xi,x,y)}(v)\sum_{n=1}^{\infty}\gamma^n\left(\rho_{[\xi]_n}(v)\right)\,\lambda^{n-1}\left(\rho_{[\xi]_{n-1}}(x)\right)\,\lambda'\left(\rho_{[\xi]_n}(v)\right)\\ & +\sum_{n=1}^{\infty}\gamma^n\left(\rho_{[\xi]_n}(v)\right)\,W_{n-1}\left(\rho_{[\xi]_{n-1}}(x)\right)\,\lambda'\left(\rho_{[\xi]_n}(v)\right).\end{aligned}$$

Clearly, $\left|\left(\ell^{ss}_{(\xi,x,y)}\right)'(v)\right|\leqslant\|X_3\|_{\infty}$ holds for all $v\in[0,1]$ and $(\xi,x,y)\in[0,1]^2\times\mathbb{R}$. Thus, by (42) and since $\|W_{n-1}\|_{\infty}$, $\|\lambda^{n-1}\|_{\infty}$ and $|\lambda'|_{\mathrm{pw},\alpha}$ are uniformly bounded with respect to n, we can obtain a bounded of the α-Hölder seminorm of the above expression by applying the triangle inequality several times. $\square$

The next simple geometrical result originates in a work of Marstrand [16].

Proposition 4.3. *For $M>0$ there is a $D_M>0$ such that*

$$\operatorname{diam}\left\{\theta\in[-M,M]:\left|L^{\theta}_{(x,y)}(x')-y'\right|\leqslant r\right\}\leqslant\frac{D_M\,r}{\|(x,y)-(x',y')\|}$$

for all (x,y), $(x',y')\in[0,1]\times\mathbb{R}$ and $r>0$.

Proof. The proof is elementary. ☐

We continue the proof of the lower estimate. Observe that we have

$$\begin{aligned}\left|L^{\Theta(\xi,v)}_{(v,W(v))}(x)-W(x)\right| &\leqslant \left|L^{\Theta(\xi,v)}_{(v,W(v))}(x)-\ell^{ss}_{(\xi,v,W(v))}(x)\right|+\left|\ell^{ss}_{(\xi,v,W(v))}(x)-W(x)\right|\\ &\leqslant C_{\|W\|_\infty}|v-x|^{1+\alpha}+r\\ &\leqslant C_{\|W\|_\infty}\|(v,W(v))-(x,W(x))\|^{1+\alpha}+r\\ &\leqslant C_{\|W\|_\infty}(r^t)^{1+\alpha}+r\\ &\leqslant (C_{\|W\|_\infty}+1)\,r\end{aligned}$$

by Proposition 4.2, whenever both $(v,W(x))\in B_{r^t}(x,W(x))$ and $\left|\ell^{ss}_{(\xi,v,W(v))}(x)-W(x)\right|\leqslant r$ are satisfied. Thus we have

$$\begin{aligned}&\left\{\xi\in[0,1]:v,x\in G_\xi \text{ and } \left|\ell^{ss}_{(\xi,v,W(v))}(x)-W(x)\right|\leqslant r\right\}\\ \subseteq\ &\left\{\xi\in[0,1]:v,x\in G_\xi \text{ and } \left|L^{\Theta(\xi,v)}_{(v,W(v))}(x)-W(x)\right|\leqslant (C_{\|W\|_\infty}+1)\,r\right\}\\ \subseteq\ &\left\{\xi\in[0,1]: \begin{array}{l}\mathbb{P}_v\left(B_\eta\left(\Theta(\xi,v)\right)\right)\leqslant E\,\eta^{1-2\delta} \text{ for } \forall\eta>0, \text{ and}\\ \left|L^{\Theta(\xi,v)}_{(v,W(v))}(x)-W(x)\right|\leqslant (C_{\|W\|_\infty}+1)\,r\end{array}\right\}\end{aligned}$$

for $r>0$ and $v,x\in[0,1]$ such that $(v,W(v))\in B_{r^t}(x,W(x))$. Recall that $a:=\{1,\gamma_1+\gamma_2\}\in[0,1]$. For those r,v,x we have therefore

$$\begin{aligned}&\nu^-\left\{\xi\in[0,1]:v,x\in G_\xi \text{ and } \left|\ell^{ss}_{(\xi,v,W(v))}(x)-W(x)\right|\leqslant r\right\}\\ \leqslant\ &\nu^-\left\{\xi\in[0,1]: \begin{array}{l}\mathbb{P}_v\left(B_\eta\left(\Theta(\xi,v)\right)\right)\leqslant E\,\eta^{1-2\delta} \text{ for } \forall\eta>0 \text{ and}\\ \left|L^{\Theta(\xi,v)}_{(v,W(v))}(x)-W(x)\right|\leqslant (C_{\|W\|_\infty}+1)\,r\end{array}\right\}\\ =\ &\mathbb{P}_v\left\{\theta\in[-\|\Theta\|_\infty,\|\Theta\|_\infty]: \begin{array}{l}\mathbb{P}_v\left(B_\eta(\theta)\right)\leqslant E\,\eta^{1-2\delta} \text{ for } \forall\eta>0 \text{ and}\\ \left|L^{\theta}_{(v,W(v))}(x)-W(x)\right|\leqslant (C_{\|W\|_\infty}+1)\,r\end{array}\right\}\\ \leqslant\ &\mathbb{P}_v\left\{\cdots\right\}^a\\ \leqslant\ &E^a\left(\frac{D_{\|\Theta\|_\infty}(C_{\|W\|_\infty}+1)\,r}{\|(v,W(v))-(x,W(x))\|}\right)^{a(1-2\delta)} \quad =: \frac{Q\,r^{a(1-2\delta)}}{\|(v,W(v))-(x,W(x))\|^{a(1-2\delta)}}\end{aligned}$$

by Proposition 4.3. Now, there is a $Q'>0$ such that for all $r>0$ and $(\xi,x)\in G$ we have

$$\begin{aligned}&\int_{\{\xi\in[0,1]:(\xi,x)\in G\}} b(\xi,x,r,t)\,d\nu^-(\xi)\\ \leqslant\ &Q\,r^{a(1-2\delta)}\int_{B_{r^t}(x,W(x))}\frac{d\mu(v,y)}{\|(v,y)-(x,W(x))\|^{a(1-2\delta)}}\\ \leqslant\ &Q\,r^{a(1-2\delta)}\sum_{n\geqslant\lfloor -t\log r\rfloor}\int_{B_{e^{-n}}(x,W(x))\setminus B_{e^{-n-1}}(x,W(x))}\frac{d\mu(v,y)}{\|(v,y)-(x,W(x))\|^{a(1-2\delta)}}\\ \leqslant\ &Q\,r^{a(1-2\delta)}\sum_{n\geqslant\lfloor -t\log r\rfloor}\frac{\mu\left(B_{e^{-n}}\left(x,W(x)\right)\right)}{e^{(-n-1)a(1-2\delta)}}\\ \leqslant\ &Q\,r^{a(1-2\delta)}\sum_{n\geqslant\lfloor -t\log r\rfloor}\frac{E\left(e^{-n}\right)^{\gamma_1+\gamma_2-a\delta}}{e^{(-n-1)a(1-2\delta)}}\\ \leqslant\ &Q'\,r^{a(1-2\delta)+t(\gamma_1+\gamma_2-a+a\delta)}\\ \leqslant\ &Q'\,r^{a(1-2\delta)+t(\gamma_1+\gamma_2-a)}.\end{aligned}$$

As the integration by ν yields

$$\int_G b(\xi,x,r,t)\,d(\nu^-\otimes\nu)(\xi,x)\leqslant Q'\,r^{a(1-2\delta)+t(\gamma_1+\gamma_2-a)},$$

we can derive by Fatou lemma

$$\int_G \liminf_{r\to 0} \frac{b(\xi,x,r,t)}{r^{a(1-2\delta)+t(\gamma_1+\gamma_2-a)}}\, d(\nu^-\otimes\nu)(\xi,x) \leqslant Q',$$

which in turn implies

$$a(1-2\delta)+t(\gamma_1+\gamma_2-a) \leqslant \limsup_{r\to 0} \frac{\log b(\xi,x,r,t)}{\log r} \tag{43}$$

for $\nu^-\otimes\nu$-a.a. $(\xi,x)\in G$.

4.3. **Upper estimate.** The goal of this subsection is to prove the upper estimate of (40).

We can chose $\varepsilon_1>0$ so that both

$$G' := \left\{(\xi,x)\in G : \begin{array}{l} \mu\left(B_r\left(x,W(x)\right)\cap G_\xi\times\mathbb{R}\right)\geqslant r^{\gamma_1+\gamma_2+\delta} \text{ and} \\ \nu_{(\xi,q_\xi(x))}\left(B_r(x)\right)\leqslant r^{\gamma_1-\delta} \end{array} \text{ for } \forall r\in(0,\varepsilon_1)\right\}$$

and

$$G'' := \left\{(\xi,x)\in G' : \begin{array}{r} \nu_{(\xi,q_\xi(x))}\left(B_r(x)\cap G'_\xi\cap (W-\ell^{ss}_{(\xi,x,W(x))})^{-1}(0)\right)\geqslant r^{\gamma_1+\delta} \\ \text{for } \forall r\in(0,\varepsilon_1^{\frac{1}{1+\alpha}}) \end{array}\right\}$$

have positive $\nu^-\otimes\nu$ measures in view of (38), (39) and Lemma 2.12 together with the fact that by (19) we have

$$\nu^-\otimes\nu\left\{(\xi,x)\in[0,1]^2 : \nu_{(\xi,q_\xi(x))}\left((W-\ell^{ss}_{(\xi,x,W(x))})^{-1}(0)\right)=1\right\}=1.$$

Let $(\xi,x)\in G''$ and $r\in(0,\varepsilon_1)$ be fixed.

By Vitali covering theorem there are $N_r\in\mathbb{N}$ and $x_1,\dots,x_{N_r}\in G'_\xi\cap B_{(r/2)^t}(x)\cap(W-\ell^{ss}_{(\xi,x,W(x))})^{-1}(0)$ such that

$$G'_\xi\cap B_{(r/2)^t}(x)\cap(W-\ell^{ss}_{(\xi,x,W(x))})^{-1}(0)\subseteq\bigcup_{i=1}^{N_r}B_r(x_i)$$

with $B_{r/3}(x_1),\dots,B_{r/3}(x_{N_r})$ being disjoint. As we have $\nu_{(x,q_\xi(x))}=\nu_{(x_i,q_\xi(x_i))}$ for $i=1,\dots,N_r$, taking the measure $\nu_{(x,q_\xi(x))}$ of both sides of the above inclusion yields

$$\left(\frac{r}{2}\right)^{t(\gamma_1+\delta)}\leqslant\sum_{i=1}^{N_r}r^{\gamma_1-\delta},$$

i.e.

$$N_r\geqslant 2^{-t(\gamma_1+\delta)}r^{(t-1)\gamma_1+2\delta}. \tag{44}$$

On the other hand, by Proposition 2.4 there are $c\in(0,1/3)$ and $\varepsilon_2\in(0,1)$ depending on t and $\|X_3\|_\infty$, such that $B_{cr}\left(u,\ell^{ss}_{(\xi,x,W(x))}(u)\right)\subseteq\Sigma_r(\xi,x)\cap B_{r^t}(x,W(x))$ holds for all $r\in(0,\varepsilon_2)$ and $u\in B(x,(r/2)^t)$. In particular, we have

$$B_{c\,r}(x_i,W(x_i))\subseteq\Sigma_r(\xi,x)\cap B_{r^t}(x,W(x)) \tag{45}$$

for $r\in(0,\varepsilon_2)$ and $i=1,\dots,N_r$. As the balls $B_{cr}\left(x_1,W(x_1)\right),\dots,B_{cr}\left(x_{N_r},W(x_{N_r})\right)$ are disjoint, from (44) and (45) follows

$$\begin{aligned} b(\xi,x,r,t) &= \mu\left(G_\xi\times\mathbb{R}\cap\Sigma_r(\xi,x)\cap B_{r^t}(x,W(x))\right) \\ &\geqslant \mu\left(\bigcup_{i=1}^{N_r}B_{cr}\left(x_i,W(x_i)\right)\cap G_\xi\times\mathbb{R}\right) \\ &\geqslant N_r(cr)^{\gamma_1+\gamma_2+\delta} \\ &\geqslant c^{\gamma_1+\gamma_2+\delta}2^{-t(\gamma_1+\delta)}r^{(t-1)\gamma_1+\gamma_1+\gamma_2+3\delta} \\ &:= Q'''\,r^{t\gamma_1+\gamma_2+3\delta}. \end{aligned}$$

Consequently, we have

$$\limsup_{r\to 0} \frac{\log b(\xi,x,r,t)}{\log r} \leqslant t\gamma_1+\gamma_2+3\delta. \tag{46}$$

for all $(\xi,x)\in G''$.

5. Hausdorff dimension of $\Theta(\cdot,x)$

The assumption of Theorems 2 and 3 that the distribution of $\Theta(\cdot,x)$ under ν^- has Hausdorff dimension 1 for ν-a.a. $x\in[0,1]$ is generally of course not true, see also Remark 1.5. We study the condition for a simple case that is relevant to Theorem 4 and 5. For a given probability vector $\mathbf{p}=(p_i)_{i=0}^{\ell-1}$ we define the Bernoulli measure $\nu_{\mathbf{p}}\in\mathcal{P}([0,1])$ as the τ-invariant measure determined by

$$\nu_{\mathbf{p}}(I_N(x)) := \prod_{i=0}^{N} p_{k(\tau^i(x))}$$

for all $x\in[0,1]$ and $N\in\mathbb{N}$. Clearly, Bernoulli measures are Gibbs measures. In addition, they satisfy $\nu_{\mathbf{p}}^-=\nu_{\mathbf{p}}$ and $\nu_{\mathbf{p}}^{\mathrm{ext}}=\nu_{\mathbf{p}}\otimes\nu_{\mathbf{p}}$.

Henceforth we consider only the Bernoulli measure $\nu_{\mathbf{p}}$, i.e. we consider the distribution of $\Theta(x,\cdot)$ under $\nu_{\mathbf{p}}$ for $\nu_{\mathbf{p}}$-a.a. $x\in[0,1]$. In fact, under the assumption of Theorem 4 or 5 the equilibrium measure $\nu_{\tau,\lambda}$ associated with the Bowen equation (1) is a Bernoulli measure as the following lemma says.

Lemma 5.1. *Under the assumption of Theorem 4 or 5, $\nu_{\tau,\lambda}$ is the Bernoulli measure with $\mathbf{p}=(|I_i|^{s(\tau,\lambda)}\gamma_i^{-1})_{i=0}^{\ell-1}$, i.e. the unique τ-invariant measure satisfying $\nu_{\tau,\lambda}(I_i)=|I_i|^{s(\tau,\lambda)}\gamma_i^{-1}$ for $i\in\{0,\dots,\ell-1\}$, where $\gamma_i:=|I_i|\lambda_i$ or $\gamma_i:=|I_i|^{1-\theta}$ in case of Theorem 4 or 5, respectively.*

Furthermore, in the latter case we have $s(\tau,\lambda)=2-\theta$, and $\nu_{\mathbf{p}}$ is the Lebesgue measure.

Proof. By the canonical coding of $[0,1]$ w.r.t. $(I_i)_{i=0}^{\ell-1}$ we can apply the variational principle for the one-sided shift space with ℓ symbols. Since the pull-back of the potential $(1-s)\log\tau'+\log\lambda = s\log|I_{k(\cdot)}|-\log\gamma_{k(\cdot)}$ depends only on the first symbol, the equilibrium measure on the shift space as well as the corresponding one on $[0,1]$ are both Bernoulli measures. We can also calculate the topological pressure

$$P((1-s)\log\tau'+\log\lambda)=\log\left(\sum_{i=0}^{\ell-1}|I_i|^{s(\tau,\lambda)}\gamma_i^{-1}\right),$$

which gives the parameter of the Bernoulli measure. We refer to e.g. [2, Chapter 3] for the terminology we used and some related observations.

Finally, under the assumption of Theorem 5 the Bowen equation is $P((1-s-\theta)\log\tau')=0$. Thus $1-s-\theta=-1$ and the equilibrium state $\nu_{\tau,\lambda}$ is the Lebesgue measure. □

5.1. **Case of self-similar measure.** Here we prove Theorem 4. Suppose $\ell=2$ and τ is piecewise linear. Suppose also that λ and g satisfy $\lambda(x):=\gamma_{k(x)}\,|I_{k(x)}|$ and $g'(x)=:a_{k(x)}$ for given constants $\gamma_0,\gamma_1\in(0,1)$ and $a_0,a_1\in\mathbb{R}$.

Observe that under these conditions Θ does not depend on x, i.e. we can write $\tilde\Theta:=\Theta(\cdot,x)$ for all x. We consider the parametrisation $t\mapsto t\lambda$ and thus $W_{\tau,t\lambda}(x):=\sum_{n=0}^{\infty}t^n\lambda^n(x)g(\tau^n(x))$ for those $t\in(0,\infty)$ which satisfy $t\lambda_i\in(|I_i|,1)$ for $i=0,1$. Correspondingly, let $\tilde\Theta_t$ denote the '$\tilde\Theta$ with respect to the parameter t, i.e. $\tilde\Theta_t(\xi)=-\sum_{n=1}^{\infty}t^{-n}\gamma^n(\xi)a_{k(\xi)}$, where $\gamma(\xi):=\gamma_{k(\xi)}$.

Proposition 5.2. *Under the above assumption, if $\gamma_0a_0\neq\gamma_1a_1$, there is a set $E\subset\mathbb{R}$ of Hausdorff dimension 0 such that the distribution of $\tilde\Theta_t$ under $\nu_{\mathbf{p}}$ has Hausdorff dimension 1 for any probability vector $\mathbf{p}$ and $t\in(\max\{\gamma_0,\gamma_1\},\infty)\setminus E$ whenever*

$$h_{\nu_{\mathbf{p}}}\geqslant-\int\log\left(t^{-1}\gamma_{k(\xi)}\right)\,d\nu_{\mathbf{p}}(\xi). \tag{47}$$

Proof. We consider the parametrisation $s = \max\{\gamma_0, \gamma_1\} \cdot t^{-1}$. Observe that, for $s \in (-1, 1) \setminus \{0\}$, the distribution of $\tilde{\Theta}_{\max\{\gamma_0,\gamma_1\}\cdot s^{-1}}$ under $\nu_{\mathbf{p}}$ is a self-similar measure with respect to the probability vector $\mathbf{p}$ and the IFS $\Phi_s := \{\gamma_i(s) \cdot (x - a_i) : i = 0, 1\}$ in the sense of [7], where

$$\gamma_i(s) := \frac{\gamma_i \cdot s}{\max\{\gamma_0, \gamma_1\}}.$$

As the separation condition on Φ_s of [7, Theorem 1.8] is satisfied due to the assumption $\gamma_0 a_0 \neq \gamma_1 a_1$, by [7, Theorem 1.7] there is a $\tilde{E} \subset (-1, 1) \setminus \{0\}$ with Hausdorff dimension 0 such that the distribution of $\tilde{\Theta}_{\max\{\gamma_0,\gamma_1\}\cdot s^{-1}}$ under $\nu_{\mathbf{p}}$ has Hausdorff dimension $\min\{1, \frac{h_{\nu_{\mathbf{p}}}}{-\int \log\left(t^{-1}\gamma_{k(\xi)}\right) d\nu_{\mathbf{p}}(\xi)}\}$ for any probability vector $\mathbf{p}$ and $s \in (-1, 1) \setminus \tilde{E}$. As a locally bi-Lipshitz continuous transformation preserves the Hausdorff dimension, the claim is satisfied by letting $E := \{t \in (\max\{\gamma_0, \gamma_1\}, \infty) : \max\{\gamma_0, \gamma_1\} \cdot s^{-1} \in \tilde{E}\}$. □

Proof of Theorem 4. Let $E \subset \mathbb{R}$ be the set of Proposition 5.2. Observe that the function $W_{\tau,t\lambda}$ satisfies $(\tau')^{-1} < t\lambda < 1$, as $t \in (\max\{\gamma_0, \gamma_1\}, \infty)$. Thus by Theorem 3 and Lemma 5.1 we have $\dim_H(\mathrm{graph}(W_{\tau,t\lambda})) = \dim_B(\mathrm{graph}(W_{\tau,t\lambda})) = s(\tau, t\lambda)$ for all $t \in (\max\{\gamma_0, \gamma_1\}, \infty) \setminus E$ whenever the condition (47) is satisfied. Now we shall check it for each $t \in \left(\max\{\gamma_0, \gamma_1\}, \min\{\frac{\gamma_0}{\sqrt{|I_0|}}, \frac{\gamma_1}{\sqrt{|I_1|}}\}\right]$. Recall that $P((1 - s(\tau, t\lambda)) \log \tau' + \log(t\lambda)) = 0$ by the definition. In view of $(\lambda\tau')^{-1} = \gamma < t$ we have $P(-s(\tau, t\lambda) \log \tau') \leqslant P(-s(\tau, t\lambda) \log \tau' + \log(\tau' t\lambda)) = 0$, which implies $s(\tau, t\lambda) \geqslant 1$. Hence by Proposition 2.2, from the equilibrium expression (or from (2)) follows

$$\begin{aligned}
h_{\nu_{\nu_{\tau,t\lambda}}} &= (s(\tau, t\lambda) - 1) \int \log \tau' d\nu_{\tau,t\lambda} - \int \log(t\lambda)\, d\nu_{\tau,t\lambda} \\
&\geqslant -\int \log(t\lambda)\, d\nu_{\tau,t\lambda} \\
&= -\int \log\left(t^{-1}\gamma\right) d\nu_{\tau,t\lambda} + \int \log\left(\gamma/(t^2\lambda)\right) d\nu_{\tau,t\lambda} \\
&= -\int \log\left(t^{-1}\gamma\right) d\nu_{\tau,t\lambda} + \int \log\left(\gamma_{k(\cdot)}^2/(t^2|I_{k(\cdot)}|)\right) d\nu_{\tau,t\lambda} \quad \geqslant \quad -\int \log\left(t^{-1}\gamma\right) d\nu_{\tau,t\lambda},
\end{aligned}$$

since $\gamma/\lambda = \gamma^2\tau'$ and $t \leqslant \min\{\frac{\gamma_0}{\sqrt{|I_0|}}, \frac{\gamma_1}{\sqrt{|I_1|}}\}$. □

5.2. **Sufficient condition through transversality.** In [22] Tsujii introduced (ε, δ)-transversality to study the L^2-absolute continuity of a sort of SRB-measures that corresponds the distribution of Θ of this note. This relation is pointed out in [1]. In order to check the condition of Theorem 5 we develop his method. Thus we assume the setting of that theorem. In particular, let $\ell \geqslant 2$, let τ be piecewise linear and $\lambda := (\tau')^{-\theta}$ for a $\theta \in (0, 1)$. Note, however, that g does not need to be the specific functions as in that theorem until we require it explicitly.

As already proved in Lemma 5.1 the measure $\nu_{\tau,\lambda}$ is nothing but the Lebesgue measure m on $[0, 1]$. Thus $\nu_{\tau,\lambda} = \nu_{\mathbf{p}_c}$ for the critical probability vector $\mathbf{p}_c := (|I_0|, \ldots, |I_{\ell-1}|)$.

Observe that $\lambda(x) = \lambda_{k(x)}$ and $\gamma(x) = (\tau'\lambda)^{-1}(x) = \gamma_{k(x)}$, where $\lambda_i := |I_i|^\theta$ and $\gamma_i := |I_i|^{1-\theta}$ for $i = 0, \ldots, \ell - 1$. In particular, $\gamma(\rho_{k(\xi)}(x)) = \gamma(\xi)$. Thus $\Theta(\xi, x) = \sum_{n=1}^{\infty} \gamma^n(\xi)\, g'(\rho_{[\xi]_n}(x))$ is differentiable so that the following consideration makes sense.

Let $\varepsilon, \delta > 0$ and $\xi, \eta \in [0, 1]$. We say that $\Theta(\xi, \cdot)$ and $\Theta(\eta, \cdot)$ are (ε, δ)-transversal, if for each $x \in [0, 1]$ holds either

$$|\Theta(\xi, x) - \Theta(\eta, x)| > \varepsilon \quad \text{or} \quad \left|\frac{\partial\Theta}{\partial x}(\xi, x) - \frac{\partial\Theta}{\partial x}(\eta, x)\right| > \delta.$$

Observe that the distribution of $(\xi, x) \mapsto (x, \Theta(\xi, x))$ under $\nu_{\mathbf{p}} \otimes \nu_{\mathbf{p}}$ is an invariant ergodic measure of the dynamical system $f : [0, 1] \times \mathbb{R} \to [0, 1] \times \mathbb{R}$ defined by

$$f(x, y) := (\tau(x),\ \gamma(x) \cdot (y - g'(x))).$$

Indeed, we have the invariance

$$f(x, \Theta(\xi, x)) = \left(\tau(x), \Theta \circ B^{-1}(\xi, x)\right).$$

Let $\zeta_{\mathbf{p}} := \nu_{\mathbf{p}} \otimes \nu_{\mathbf{p}} \circ (\mathrm{Id}, \Theta)^{-1}$ and $\zeta_{\mathbf{p},x} := \nu_{\mathbf{p}} \circ \Theta(\cdot, x)^{-1}$ so that $\zeta_{\mathbf{p}} = \int \delta_{\{x\}} \times \zeta_{\mathbf{p},x}\, d\nu_{\mathbf{p}}(x)$. Slightly abusing the notation, we define

$$f\zeta_{\mathbf{p},x}(A) := \zeta_{\mathbf{p},x} \circ f(x, \cdot)^{-1}.$$

Furthermore, let m denote (also) the Lebesgue measure on $\mathbb{R}$. For $i, j \in \{0, \dots, \ell - 1\}$ and $r > 0$ we define

$$\begin{aligned} I_{\mathbf{p}}(r) &:= \frac{1}{r^2} \int_{[0,1]} \|\zeta_{\mathbf{p},x}\|_r^2 \, d\nu_{\mathbf{p}}(x), \text{ and} \\ I_{\mathbf{p}}(r; i, j) &:= \frac{1}{r^2} \int_{[0,1]} \left(f\zeta_{\mathbf{p},\rho_i(x)}, f\zeta_{\mathbf{p},\rho_j(x)} \right)_r \, d\nu_{\mathbf{p}}(x), \end{aligned}$$

where

$$(\nu, \tilde{\nu})_r := \int_{\mathbb{R}} \nu(B_r(z))\, \tilde{\nu}(B_r(z))\, dm(z)$$

and $\|\nu\|_r^2 := (\nu, \nu)_r$ for $\nu, \tilde{\nu} \in \mathcal{P}(\mathbb{R})$ and $r > 0$.

Proposition 5.3. *We have*

$$\zeta_{\mathbf{p},x}(A) = \sum_{i=0}^{\ell-1} p_i\, f\zeta_{\mathbf{p},\rho_i(x)}(A)$$

for all $A \in \mathcal{B}(\mathbb{R})$.

Proof. For $A \in \mathcal{B}(\mathbb{R})$ we have

$$\begin{aligned} \zeta_{\mathbf{p},x}(A) &= \nu_{\mathbf{p}}(\{\xi \in [0,1] : \Theta(\xi, x) \in A\}) \\ &= \nu_{\mathbf{p}}\left(\left\{\xi \in [0,1] : \Theta \circ B^{-1}\left(\tau(\xi), \rho_{k(\xi)}(x)\right) \in A\right\}\right) \\ &= \sum_{i=0}^{\ell-1} \nu_{\mathbf{p}}\left(\left\{\xi \in I_i : \Theta \circ B^{-1}\left(\tau(\xi), \rho_i(x)\right) \in A\right\}\right) \\ &= \sum_{i=0}^{\ell-1} p_i\, \nu_{\mathbf{p}}(\{\xi \in [0,1] : \Theta \circ B^{-1}(\xi, \rho_i(x)) \in A\}) \\ &= \sum_{i=0}^{\ell-1} p_i\, \nu_{\mathbf{p}}\left(\{\xi \in [0,1] : f(\rho_i(x), \Theta(\xi, \rho_i(x))) \in \{x\} \times A\}\right) \\ &= \sum_{i=0}^{\ell-1} p_i\, \zeta_{\mathbf{p},\rho_i(x)}(\{y \in \mathbb{R} : f(\rho_i(x), y) \in \{x\} \times A\}) \\ &= \sum_{i=0}^{\ell-1} p_i\, f\zeta_{\mathbf{p},\rho_i(x)}(A). \end{aligned}$$

□

Proposition 5.4. *Let $0 \leqslant i < j \leqslant \ell - 1$. If $\Theta(\xi, \cdot)$ and $\Theta(\eta, \cdot)$ are (ε, δ)-transversal for all $(\xi, \eta) \in I_i \times I_j$, then we have*

$$I_{\mathbf{p}_c}(r; i, j) \leqslant 8\delta^{-1} \max\{4\alpha/\varepsilon, 1\}$$

for all $r \in (0, \varepsilon/4)$, where $\alpha := \|\frac{\partial \Theta}{\partial x}\|_\infty$.

Proof. Observe $I_{\mathbf{p}_c}(r; i, j) = \frac{1}{r^2} \int \left(f\zeta_{\mathbf{p},\rho_i(x)}, f\zeta_{\mathbf{p},\rho_j(x)} \right)_r \, dm(x)$. As the integral part is bounded by $8\delta^{-1} r^2 \max\{4\alpha/\varepsilon, 1\}$ analogously to [22, Proposition 6], the claim follows. □

Proposition 5.5. *We have*

$$\|f\zeta_{\mathbf{p},\rho_i(x)}\|_r^2 = \gamma(\rho_i(x)) \cdot \|\zeta_{\mathbf{p},\rho_i(x)}\|_{r/\gamma(\rho_i(x))}^2$$

for all $r > 0$, $x \in [0,1]$ and $i \in \{0, \dots, \ell - 1\}$.

Proof. As

$$f\zeta_{\mathbf{p},\rho_i(x)}\left(B_r(z)\right) = \zeta_{\mathbf{p},\rho_i(x)}\left(B_{r/\gamma(\rho_i(x))}\left(\frac{z}{\gamma(\rho_i(x))} + g'\left(\rho_i(x)\right)\right)\right),$$

the claim follows by the substitution formula of the integral. □

Proposition 5.6. *Suppose that $\Theta(\xi,\cdot)$ and $\Theta(\eta,\cdot)$ are (ε,δ)-transversal for all $(\xi,\eta)\in I_i\times I_j$ and $0\leqslant i<j\leqslant \ell-1$. Then we have*

$$I_{\mathbf{p}_c}(r) \leqslant \beta\, I_{\mathbf{p}_c}\left(\frac{r}{\min_{[0,1]}\gamma}\right) + 8\delta^{-1}\max\{4\alpha/\varepsilon, 1\}$$

for all $k\in\mathbb{N}_0$ and $r\in(0,\varepsilon/4)$, where $\alpha := \|\frac{\partial\Theta}{\partial x}\|_\infty$ and $\beta := \frac{\max_i \frac{|I_i|^2}{\lambda_i}}{(\min_{[0,1]}\gamma)^2}$.

Proof. Let $\mathbf{p}=\mathbf{p}_c$. By Propositions 5.3 and 5.4 we have

$$\begin{aligned} I_{\mathbf{p}}(r) &= \sum_{i,j=0}^{\ell-1} p_i p_j\, I_{\mathbf{p}}(r;i,j) \\ &\leqslant \sum_{i=0}^{\ell-1} p_i p_i\, I_{\mathbf{p}}(r;i,i) + \sum_{i\neq j} p_i p_j\, I_{\mathbf{p}}(r;i,j) \\ &\leqslant \sum_{i=0}^{\ell-1} p_i p_i\, I_{\mathbf{p}}(r;i,i) + 8\delta^{-1}\max\{4\alpha/\varepsilon,1\}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \sum_{i=0}^{\ell-1} p_i p_i\, I_{\mathbf{p}}(r;i,i) &= \sum_{i=0}^{\ell-1} \frac{p_i^2}{r^2}\int \gamma(\rho_i(x))\cdot \|\zeta_{\mathbf{p},\rho_i(x)}\|^2_{r/\gamma(\rho_i(x))}\, dm(x) \\ &= \sum_{i=0}^{\ell-1} \frac{p_i^2}{r^2}\int_{I_i} \gamma(x)\cdot \|\zeta_{\mathbf{p},x}\|^2_{r/\gamma(x)}\cdot \tau'(x)\, dm(x) \\ &\leqslant \sum_{i=0}^{\ell-1} \frac{\max_i \frac{|I_i|^2}{\lambda_i}}{(\min_{[0,1]}\gamma)^2}\left(\frac{\min_{[0,1]}\gamma}{r}\right)^2 \int_{I_i} \|\zeta_{\mathbf{p},x}\|^2_{r/(\min_{[0,1]}\gamma)} dm(x) \\ &\leqslant \beta\, I_{\mathbf{p}}\left(\frac{r}{\min_{[0,1]}\gamma}\right) \end{aligned}$$

by Proposition 5.5. Thus the claim is proved. □

Lemma 5.7. *Under the assumption of Proposition 5.6, if $\beta<1$, then $\liminf_{r\to 0} I_{\mathbf{p}_c}(r)<\infty$. In this case, the distribution of $\Theta(\cdot,x)$ under $\nu_{\mathbf{p}_c}$ has a conditional L^2-densities h_x w.r.t. m for $\nu_{\mathbf{p}_c}$-a.a. x such that $\int \|h_x\|_2^2\, d\nu_{\mathbf{p}_c}(x)<\infty$.*

In particular, the distribution of $\Theta(\cdot,x)$ under $\nu_{\mathbf{p}_c}$ has Hausdorff dimension 1 for $\nu_{\mathbf{p}_c}$-a.a. x.

Proof. For example, we can take a sequence $r_k := \varepsilon\,(\min_{[0,1]}\gamma)^k/8$ so that

$$I_{\mathbf{p}}(r_k) \leqslant \beta^k I(\varepsilon/8) + \frac{8\delta^{-1}\max\{4\alpha/\varepsilon,1\}}{1-\beta}.$$

Since we have thus $\liminf_{r\to 0} I_{\mathbf{p}}(r)<\infty$, the remaining part of the fist claim can be concluded analogously to the proof of [22, Corollary 5].

Finally, the absolute continuity to Lebesgue measure implies the full Hausdorff dimension due to Lebesgue differentiation theorem. □

Now, we consider the transversality in case $g(x) = \cos(2\pi x)$. We extend an idea of [1] to find explicit parameters. Observe we have

$$\begin{aligned}
\Theta(\xi, x) &= -2\pi \sum_{n=1}^{\infty} \gamma^n(\xi) \sin\left(2\pi \rho_{[\xi]_n}(x)\right), \text{ and} \\
\frac{\partial \Theta}{\partial x}(\xi, x) &= (2\pi)^2 \sum_{n=1}^{\infty} \left(\frac{\gamma}{\tau'}\right)^n (\xi) \cos\left(2\pi \rho_{[\xi]_n}(x)\right).
\end{aligned}$$

Lemma 5.8. *Suppose that $g(x) = \cos(2\pi x)$. If*

$$G(\min\gamma, \max\gamma) + G(\min\frac{\gamma}{\tau'}, \max\frac{\gamma}{\tau'}) < \delta_0,$$

then there is a $\delta > 0$ such that $\Theta(\xi, \cdot)$ and $\Theta(\eta, \cdot)$ are (δ, δ)-transversal for all $(\xi, \eta) \in I_i \times I_j$ and $0 \leqslant i < j \leqslant \ell - 1$, where δ_0 and G are defined in (5).

Proof. Let $\xi \in I_i$ and $\eta \in I_j$ for some $0 \leqslant i < j \leqslant \ell - 1$. From

$$\begin{aligned}
\gamma_i \sin(2\pi\rho_i(x)) - \gamma_j \sin(2\pi\rho_j(x)) &= (\gamma_i - \gamma_j) \sin\left(2\pi\frac{\rho_i(x) + \rho_j(x)}{2}\right) \cos\left(2\pi\frac{\rho_i(x) - \rho_j(x)}{2}\right) \\
&\quad + (\gamma_i + \gamma_j) \cos\left(2\pi\frac{\rho_i(x) + \rho_j(x)}{2}\right) \sin\left(2\pi\frac{\rho_i(x) - \rho_j(x)}{2}\right)
\end{aligned}$$

follows

$$\begin{aligned}
(2\pi)^{-1}|\Theta(\xi, x) - \Theta(\eta, x)| &\geqslant |\gamma_i \sin(2\pi\rho_i(x)) - \gamma_j \sin(2\pi\rho_j(x))| - 2\sum_{n=2}^{\infty} (\max\gamma)^n \\
&\geqslant (\gamma_i + \gamma_j) \left|\sin\left(2\pi\frac{\rho_i(x) - \rho_j(x)}{2}\right)\right| \left|\cos\left(2\pi\frac{\rho_i(x) + \rho_j(x)}{2}\right)\right| \\
&\quad - \frac{2(\max\gamma)^2}{1 - (\max\gamma)} - |\gamma_i - \gamma_j| \\
&\geqslant 2(\min\gamma) \left|\sin\left(2\pi\frac{\rho_i(x) - \rho_j(x)}{2}\right)\right| \left|\cos\left(2\pi\frac{\rho_i(x) + \rho_j(x)}{2}\right)\right| \\
&\quad - \frac{2(\max\gamma)^2}{1 - (\max\gamma)} - (\max\gamma - \min\gamma).
\end{aligned}$$

Furthermore, from

$$\begin{aligned}
&\gamma_i\, \rho_i'(x)\, \cos(2\pi\rho_i(x)) - \gamma_j\, \rho_j'(x)\, \cos(2\pi\rho_j(x)) \\
= \;& \left(\gamma_i\, \rho_i'(x) - \gamma_j\, \rho_j'(x)\right) \cos\left(2\pi\frac{\rho_i(x) + \rho_j(x)}{2}\right) \cos\left(2\pi\frac{\rho_i(x) - \rho_j(x)}{2}\right) \\
&\quad + \left(\gamma_i\, \rho_i'(x) + \gamma_j\, \rho_j'(x)\right) \sin\left(2\pi\frac{\rho_i(x) + \rho_j(x)}{2}\right) \sin\left(2\pi\frac{\rho_i(x) - \rho_j(x)}{2}\right)
\end{aligned}$$

follows

$$
\begin{aligned}
& (2\pi)^{-2}|\Theta'(\xi,x)-\Theta'(\eta,x)| \\
\geqslant\ & \left|\gamma_i\,\rho_i'(x)\,\cos(2\pi\rho_i(x))-\gamma_j\,\rho_j'(x)\,\cos(2\pi\rho_j(x))\right| - 2\sum_{n=2}^{\infty}\left(\max\frac{\gamma}{\tau'}\right)^n \\
\geqslant\ & \left(\gamma_i\,\rho_i'(x)+\gamma_j\,\rho_j'(x)\right)\left|\sin\left(2\pi\frac{\rho_i(x)-\rho_j(x)}{2}\right)\right|\left|\sin\left(2\pi\frac{\rho_i(x)+\rho_j(x)}{2}\right)\right| \\
& -\frac{2\left(\max\frac{\gamma}{\tau'}\right)^2}{1-\left(\max\frac{\gamma}{\tau'}\right)} - \left|\gamma_i(x)\,\rho_i'(x)-\gamma_j(x)\,\rho_j'(x)\right| \\
\geqslant\ & 2\left(\min\frac{\gamma}{\tau'}\right)\left|\sin\left(2\pi\frac{\rho_i(x)-\rho_j(x)}{2}\right)\right|\left|\sin\left(2\pi\frac{\rho_i(x)+\rho_j(x)}{2}\right)\right| \\
& -\frac{2\left(\max\frac{\gamma}{\tau'}\right)^2}{1-\left(\max\frac{\gamma}{\tau'}\right)} - \left(\max\frac{\gamma}{\tau'}-\min\frac{\gamma}{\tau'}\right).
\end{aligned}
$$

By squaring and summing up both equalities above we obtain

$$
\begin{aligned}
& \left(\frac{(2\pi)^{-1}|\Theta(\xi,x)-\Theta(\eta,x)| + \frac{2(\max\gamma)^2}{1-(\max\gamma)} + (\max\gamma-\min\gamma)}{2(\min\gamma)}\right)^2 \\
& + \left(\frac{(2\pi)^{-2}|\Theta'(\xi,x)-\Theta'(\eta,x)| + \frac{2\left(\max\frac{\gamma}{\tau'}\right)^2}{1-\left(\max\frac{\gamma}{\tau'}\right)} + \left(\max\frac{\gamma}{\tau'}-\min\frac{\gamma}{\tau'}\right)}{2\left(\min\frac{\gamma}{\tau'}\right)}\right)^2 \\
\geqslant\ & \sin^2\left(2\pi\frac{\rho_i(x)-\rho_j(x)}{2}\right) \quad \geqslant \delta_0.
\end{aligned}
$$

Suppose that there is a $(\omega,\eta)\in I_i\times I_j$ with $i\neq j$ such that $\Theta(\xi,\cdot)$ and $\Theta(\eta,\cdot)$ are not (δ,δ)-transversal for any $\delta>0$. Then the above inequality implies

$$
\left(\frac{\frac{2(\max\gamma)^2}{1-(\max\gamma)} + (\max\gamma-\min\gamma)}{2(\min\gamma)}\right)^2 + \left(\frac{\frac{2\left(\max\frac{\gamma}{\tau'}\right)^2}{1-\left(\max\frac{\gamma}{\tau'}\right)} + \left(\max\frac{\gamma}{\tau'}-\min\frac{\gamma}{\tau'}\right)}{2\left(\min\frac{\gamma}{\tau'}\right)}\right)^2 \geqslant \delta_0
$$

As this contradicts the assumption of the lemma, this finishes the proof. □

Proof of Theorem 5. In Lemma 5.1 we proved $s(\tau,\lambda)=2-\theta$. Thus the claim follows from Theorem 3 together with Lemmas 5.1, 5.7 and 5.8 by the insertion of $\gamma=(\tau')^\theta$ and $\gamma/\tau'=(\tau')^{\theta-1}$. □

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